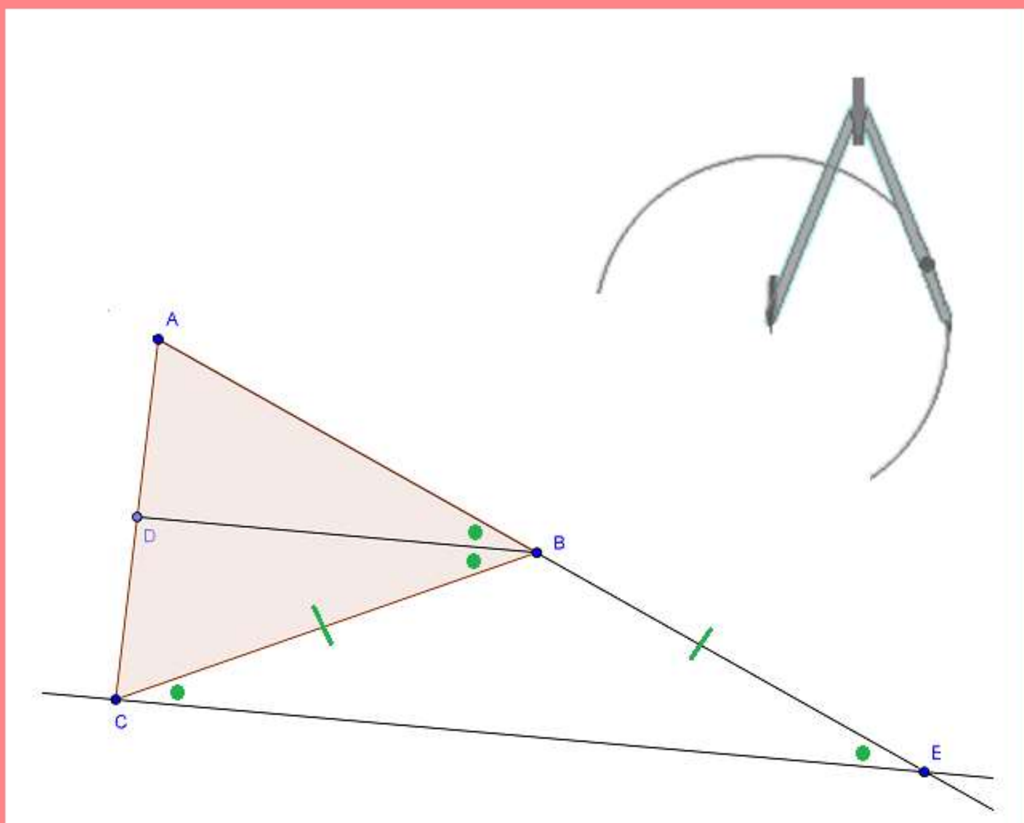
The Geometry Lab



Contents

Update Notice

Use only recently saved copies of this file. Download the latest version at http://www.guidedcourses.com.

If download is not available, or to report errors or typos in this document, contact ylani@outlook.com

Last updated: May 20, 2020.

Don't see your topic? Check the Algebra 1 topics at http://www.guidedcourses.com.

Lab Safety

Fortunately this is not the kind of lab that combines fragile glassware, dangerous chemicals, and curious young people all in the same room. This is also the reason I chose to teach math instead of chemistry. 😁

So why is it called Geometry Lab? Well, what's the use of geometry if you can't do any geometering? Or is that geometing? The word geometry actually means to measure the Earth. However, if you ask average geometry students to measure the Earth they probably won't have any idea where to start. First, we will turn our attention to measuring much smaller things, but soon we will see how Eratosthenes used geometry find the approximate circumference of the Earth around 200 BC.

Now where were we? Oh yes, lab safety. The only dangerous item in a geometry lab is a compass, which has a sharp point. If you have very young brothers or sisters you may notice that they are immediately attracted to sharp items, especially when those items are left unattended. Keep your compass in a sturdy container so you don't poke yourself with it either. Other than that, just watch out for paper cuts and you'll be fine.

In this lab, we will be using a special program called Geogebra. Geogebra allows you to let your computer do some of the work that you might find tedious and repetitive. Before you do that, make sure that you can do these same things on paper yourself. You don't want your computer to start thinking that it is smarter than you.... Actually, it's your attitude about your computer that matters. A computer is a tool, just like a protractor or a compass. Don't rely on it to do your thinking for you.

Just like in any other experiment, when you take measurements they will not be absolutely perfect, even when you use computer software. Small errors occur and add up, even when you are using Geogebra. Each experiment should be done at least 3 times, using a new figure with different dimensions each time.

Geogebra Instructions

Geogebra software is available free at https://www.geogebra.org/download. Please use the classic version if at all possible. Geogebra places an icon on your desktop or adds itself to your start screen in Windows 10. Double-click the icon to start the program. Maximize the window so you have more room. You should see 12 square buttons at the top of the screen. By hovering your cursor over each button, you can see what the button does. Only one button can be active at a time. The active button has a blue frame around it to show you that it is active. Each button has a tiny arrow in the bottom right-hand corner. Clicking on this arrow brings up a menu associated with the button.

Below the buttons, the screen is divided into two sections. The left-hand pane is called the Algebra View. If you close this, you can get it back by going to "View" at the top of the screen, and selecting Algebra. You can adjust the width of this section by dragging the line to the left or right. The right-hand pane is where you can draw. You can move the right-hand pane by using the last button. Press this button now, then click and drag the drawing area until the numbered lines are near the left and bottom of the screen, which leaves you a nice clear area for drawing. **Notice that helpful instructions appear when you hover your mouse over a button.**

Let's start by placing a point. We can do this by selecting the second button, which shows a picture of a point and its label. While this button is active, clicking on the drawing area will place points, successively labeled A, B, C, etc. Notice that these points don't just appear in the drawing area; they also show up in a list in the Algebra pane. Just like in other programs, there is an "undo" property that can be accessed by going to Edit at the top of the screen. You can use this to remove points you don't want. You can also remove points by right-clicking on them in the Algebra pane, and selecting "delete".

Once you have 3 points, you should try to connect them to make a triangle. Notice that the third button has a picture of a line, but to actually connect two points you have to access the button menu by clicking the tiny arrow in the bottom right corner. Select "segment between two points", and then place the segment by clicking on the points you want to connect.

Now that you have a triangle, you can easily change its shape by moving the points. The first button is the Move button, and it allows you to move things **only while it is active**. Practice moving the points of your triangle.

A faster way to create a triangle is by activating the Polygon button which has a picture of a triangle on it. Click to place a point, then another point, and then go back to the first point to finish your triangle. You can also change the shape of this triangle by dragging the points when the Move button is active.

Now we will label the angles in our second triangle. Click on each line, in a clockwise direction (counter-clockwise in some versions), to get the label for the inside angle. You can also click three points, again moving clockwise. This may take a little trial and error before you get used to it.

When you want to take angle measurements, sometimes you find that the point label is in your way. If so, activate the Move button and drag the point label aside, or move the angle label to where you can see it more easily.

By default, Geogebra displays its measurements to two decimal places. This measures an angle to the nearest 100th degree and line segments to the nearest tenth of a millimeter. That is nice, but your screen is not smooth. It consists of little pixels, and you'll find that this limits how accurate your measurements can be, and where exactly you can place points on your screen. Often the last two digits after the decimal point will not be accurate. You can change the number of decimal places by selecting Rounding in the Options menu. Change the setting to "1 Decimal Place", which will work best for most labs.

Once Upon a Time …

Once upon a time a long, long time ago, there lived a man named Euclid. Euclid's world (around 300 BC) was very different from our world today. People used philosophy rather than science to try to understand how things worked, and technology was limited to basic mechanical devices. However, the precursor of science and technology, mathematics, was rather well developed. Many people realized that mathematics held the promise to advance their civilization, and Euclid set out to learn everything that was known about it. He became a teacher at the ancient university of Alexandria. King Ptolemy was one of his students. When the king complained that his geometry lessons were hard, Euclid rather bluntly told him he'd have to study just like everyone else. "There is no royal road to geometry," Euclid said. There was no good mathematics textbook in those days, so Euclid organized all of the current knowledge and added his own insights and proofs to create a set of books known as "Elements".

The Elements cover various fields of mathematics, but Euclid has become most famous for his contributions to geometry. He organized geometry into a system of basic principles, and proofs based on those principles. For over 2,000 years he had the last word on the subject, and his books instructed many famous mathematicians and scientists, including Isaac Newton and Albert Einstein. The geometry that is taught in high school today is called Euclidian geometry, but students no longer read Euclid's Elements. In spite of their large size modern geometry textbooks don't seem to have room for most of the clever proofs provided by Euclid, which leaves only the conclusions for students to memorize. Geometry proofs now deal with simpler matters that can be written as "two column proofs" which are easier for teachers to correct and grade.

Einstein was so impressed with Elements that he called it the "holy little geometry book", but today's mathematicians are eager to point out flaws in it. After all, if you can prove the great Euclid wrong about even a minor point you must be very smart. To see the introduction to the original Elements without disparaging comments, read the text at http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html without clicking on any of the links. The first part seems to be the hardest to express in modern terms, so we'll take a tour through it. Read these explanations side by side with the original text:

<<Start of tour>>

Definitions

1. A point is something that is not divisible into separate parts.

The modern definition of a point is something that specifies a location, but doesn't have a size. If a point has no size it becomes difficult to see how a line is made up of points. Euclid says that a point is something that is not divisible. This implies that if a point had a size, it could not be divided into parts of an even smaller size. That is also difficult to imagine. The ancient Greeks wondered if space is infinitely divisible or not. Today we think that there may actually be a minimum length, called the Planck length, which is approximately 1.616×10^{-35} meters.

2. A line has length but no width. (A line as envisioned by Euclid can be either straight or curved in some way.)

3. The ends of a line are points.

4. A line is straight when all of the points on it line up evenly.

5. A surface only has length and width.

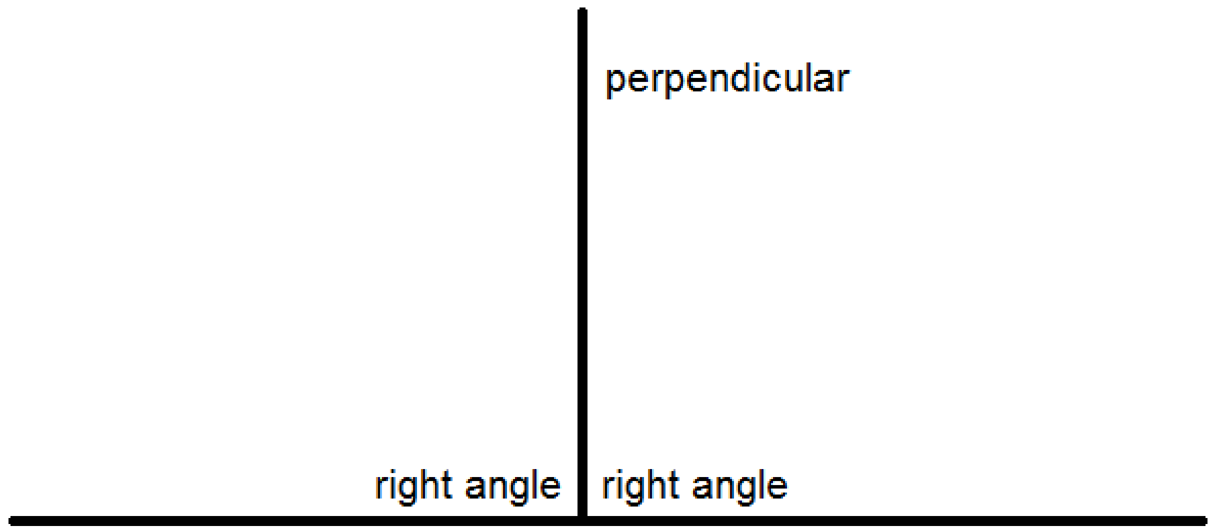
6. The edges of a surface are lines.

7. When you draw straight lines on a flat surface, the surface lines up evenly with all of them.

8. When two lines on a plane meet each other and do not form a single straight line, they create an angle. This is a plane angle (an angle in a plane). We consider a 180 degree angle to be a "straight" angle. Euclid refers to this as "two right angles".

9. When the lines containing the angle are straight lines the angle is called rectilinear. This is the kind of angle that we think of as an angle today since we do not consider angles created by the meeting of two curved lines.

10. A *perpendicular* line standing on another line creates two right angles:



11. An *obtuse angle* is an angle greater than a right angle.

12. An *acute angle* is an angle less than a right angle.

13. A *boundary* is an edge of something.

14. A *figure* is something that is contained in a boundary or boundaries.

15. A *circle* is a figure in a plane, constructed so that there is a single point from which all straight lines drawn from that point to the boundary are equal in length.

16. And that point is called the *center* of the circle.

17. A *diameter* of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and this straight line also divides the circle into two equal parts.

18. A *semicircle* is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

19. *Rectilinear figures* are those which are contained by straight lines, trilateral figures being those contained by three, *quadrilateral* those contained by four, and multilateral those contained by more than four straight lines.

20. Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides alone equal, and a *scalene triangle* that which has its three sides unequal.

21. Further, of trilateral figures, a *right-angled triangle* is that which has a right angle, an *obtuse-angled triangle* that which has an obtuse angle, and an *acute-angled triangle* that which has its three angles acute.

22. For our purposes today we need the following definitions here: Of quadrilateral figures, a *square* is that which is both equilateral (having equal sides) and right-angled, a *rectangle* that which is right-angled but not equilateral, and a *rhombus* that which is equilateral but not right-angled. A *parallelogram* is a shape with two pairs of opposite sides that are parallel.
The figure that Euclid called a rhomboid is called a parallelogram today, but our word parallelogram refers also to a square, rectangle or rhombus. In addition, we no longer use the requirements that a rectangle should not be equilateral and that a rhombus should not have right angles. This makes the square a special case of a rectangle and a special case of a rhombus.

Using the descriptions provided, make a drawing of a rhombus and a rhomboid in Geogebra. Select File -> Export, and then Graphics View as Picture. Save it as a .png file. By default the resolution is 600 dpi, which is a large high resolution picture. A word processor will scale it to the correct size for you. Alternatively, you can use the PRINT SCREEN option to take a picture of your Geogebra drawing. You can edit your picture in Microsoft Paint or Mac Paintbrush. Insert your picture into a blank word processor document. Put the title of this week's topic, "Once Upon a Time", at the top of your document as a title. If you do not have a word processor on your computer you can download a free one at http://www.openoffice.org/. (Compatible with PC or Mac. Please note that beta releases are meant to test software, so download the latest stable version.)

23. Parallel lines never meet.

Postulates

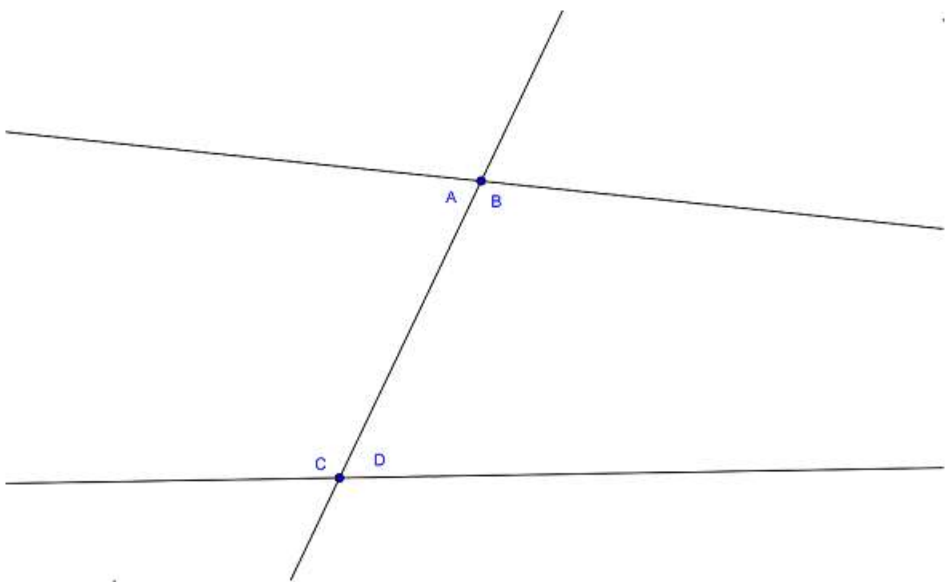
1. You can construct a *unique* straight line once you have two points (two points determine a line).

2. You can extend a line segment into a line of infinite length.

3. It is possible to construct a unique circle when you are given a center point and a radius (a center point and a radius determine a circle).

4. All right angles are equal to each other

5. This important postulate is known as the parallel postulate.



The sum of angles A and C is more than two right angles (more than 180 degrees), and the sum of B and D is less than two right angles. The two lines will meet on the right side rather than on the left. This is easy to see, but not so easy to prove.

The Postulates are things that will be assumed to be true without any proof being provided. They form a basis for constructing proofs.

The **Common Notions** are statements that are obviously true – make sure you agree, or ask for help. Euclid uses these common notions in his proofs.

1. Things which equal the same thing are also equal to each other. If a = c and b = c, then a must be equal to b.
2. If equals are added to equals, then the wholes are equal. If a = b and c = d, then a + c = b + d.
3. If equals are subtracted from equals, then the remainders are equal (c – a = d – b).
4. If you can move one thing so that it coincides exactly with another, then these two things are equal.
5. If something is made up of two or more parts, then the whole thing is larger than an individual part.

<<End of tour>>

When two planes intersect in 3D space, their intersection is a line. When two lines intersect, their intersection is a point.

Two lines that are not parallel intersect in only one single point. It makes sense that after the lines intersect they diverge, and don't come back together to intersect again. However, in geometry we try to prove things by using our basic postulates and common notions. Consider two separate lines that intersect in a point. Call the point of intersection A. In the unlikely event that they would intersect again somewhere else, we could call the second intersect point B. If the lines intersected at both points A and B, this would mean that both lines would have to pass through these two points. Use one or more of the five *Postulates* listed above to explain that this is not possible. Add your explanation to your document.

Read **Proposition 1**: http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI1.html or http://pythagoreanmath.com/euclids-elements-book-1-proposition-1/

Constructions are no longer as important to us since we have protractors and computer software. However, efforts to construct various things using only a compass (not used to measure distances) and a straightedge (something that is like a ruler with no markings) led to many important mathematical discoveries.

On paper, construct your own equilateral triangle as described in this proposition. Explain in your own words how you can be sure that you have created an equilateral triangle. Add your explanations to your word processor document. Include a picture of your paper drawing or a copy of your drawing created in Geogebra.

Make sure you have answered all questions, in complete sentences. Print a copy of your document, because computers can malfunction and lose your data. Save all your assignments for this course in a folder or binder.

Lab Reports

The next assignments that you will submit for this course will be lab reports. All lab reports should follow the format described here.

Title

Use the name of the experiment as the title for your lab report. Do not create a separate title page.

Description

Provide a brief description of what you were doing and why, so that the report makes sense without the original text describing the lab.

Results

This is the first heading. In this section, present all your results neatly. If applicable, use a table, or Tab spacing to line your results up into columns. Use a heading for each column that indicates what the column contains, for example "First Angle".

Analysis

This is the second heading. Here is where you answer the questions, and provide your own explanations for the results of the experiment. It is not necessary to copy the questions, but you should make sure your answer makes sense by itself. For example, if the question is "How does the radius of a circle relate to the diameter?" you would write "The radius is half the diameter" rather than "It is half the diameter." The last answer does not make sense without the question.

Conclusion

The conclusion should consist of a simple statement that summarizes the results. For this first lab report, your conclusion will be: The angles of a triangle appear to add up to ....

Later you will be able to quickly review the results of all of the experiments by looking at the conclusions.

The Magic Triangle

To do this experiment, you need to know something about angles and how to measure them. Read about angles on this page: http://www.mathsisfun.com/geometry/degrees.html. Answer the questions to check your understanding. Then continue to the next page, http://www.mathsisfun.com/angles.html. Click on each angle type in the table. Answer the questions here also. The answers to the questions on these pages should not be included in your lab report.

Learn how to measure angles here: http://math.about.com/library/blmeasureangles.htm. Then practice at this site: http://www.mathplayground.com/measuringangles.html. To measure angles in Geogebra, use button 8 (the 8th button from the left).

Materials

Protractor
Ruler
Paper
Geogebra
Scissors

Procedure

Draw a large triangle on paper. Carefully measure each angle inside the triangle using your protractor and record your measurements.

Draw a large triangle using the polygon button in Geogebra (see Geogebra Instructions). Click on the angle button (Button 8, counting from the left) and then on the triangle to display angle measurements (or mark each angle separately depending on your version of Geogebra). Record the measurements of each angle of the triangle. Use the move button to drag the numbers to a better spot if you have trouble reading them.

Move the points of your triangle around so you get a different triangle. Record the measurements of the angles for this triangle. Repeat this twice so that you have measurements for 4 triangles altogether.

For each triangle you drew, find the sum of the measurements of its three angles.

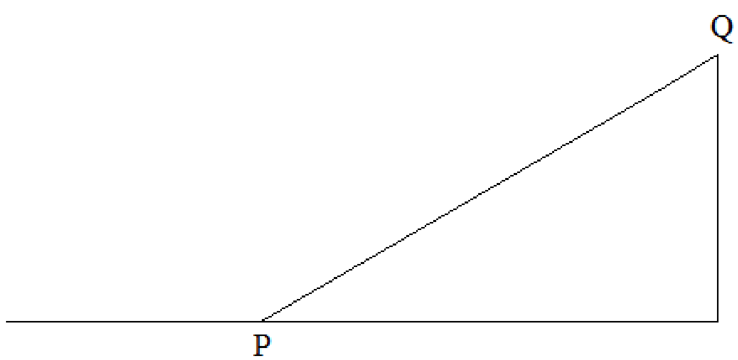
Analysis

What is the sum of the angles of your triangles, to the nearest degree?

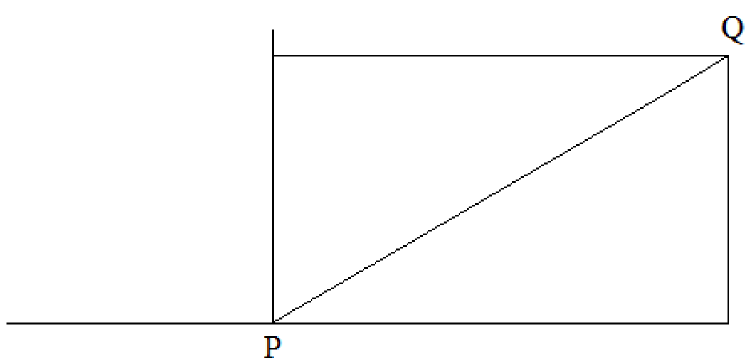
Do you think that the sum is the same for any triangle?

Hmmm, why is this? How does the triangle “know” to always have the same sum for its angles? Ooh, it's magic!

Let's take the simplest possible magic triangle, which is one that has a right (90 degree) angle, and put it in a box. Create a large rectangle on paper, as explained below.



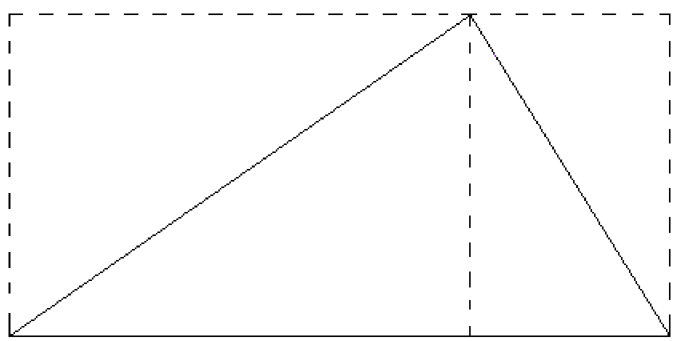
Extend the side at point P so it looks like the picture above to make it easier to line up your protractor. Next, use the protractor to create a perpendicular line at point P. Perpendicular lines are at a 90 degree angle to each other, like the lines of a "+" sign. Do the same at point Q so you have a nice rectangle.



What is the sum of the angles of a rectangle? A square?

Your rectangle is composed of two triangles. Are these two triangles similar to each other? Take some scissors and carefully cut out both triangles so you can compare their angles. Which angles are the same? If you find an angle on one triangle that is identical to an angle on the other triangle, mark both with a colored dot. Use a different color for each pair of angles that are the same. Put the rectangle back together and take a picture to show where you put the dots. Add the picture to your report.

Next we will consider a random triangle that does not have a 90 degree angle, and put it in a box. Draw a random triangle on paper now. Use your protractor to construct a line through the point of the triangle that is closest to the top, perpendicular to the line that is across from this point, closest to the bottom. This line divides your triangle into two triangles that have 90 degree angles. Construct rectangles around them like you did before.



It would seem reasonable to think that the 180 degree angle sum of a triangle can somehow be traced back to the 360 angle sum of a rectangle, but how? Save this last drawing for later, and let's follow the magic triangle down the rabbit hole...

(If you have not read Lewis Carroll's Alice in Wonderland, about a girl following a talking rabbit into a hole, download it here: http://www.gutenberg.org/ebooks/11.)

"Vertical" Angles

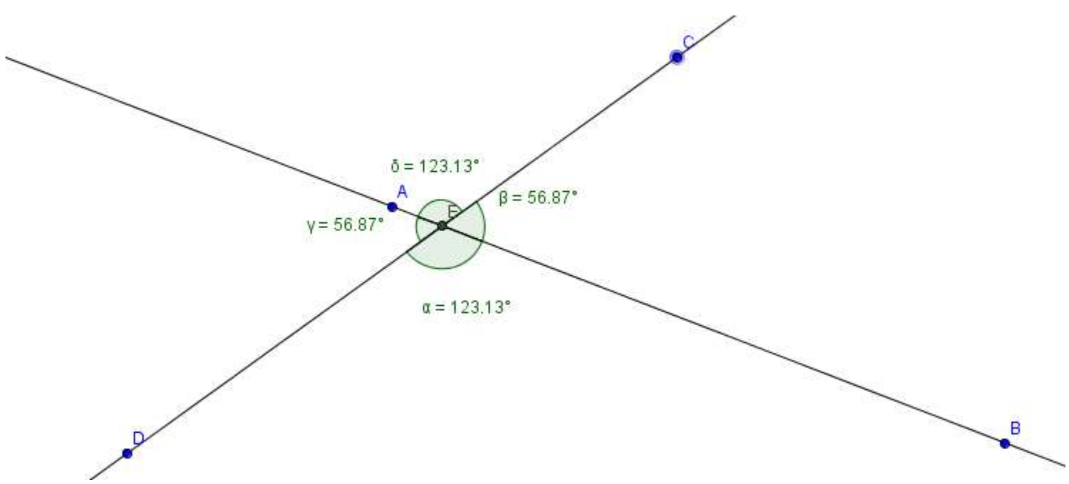
Materials

Protractor
Ruler
Paper
Geogebra

Procedure

Take a blank sheet of paper. At the top edge, make two marks at a random distance from each other. Do the same at the bottom edge. Now use your ruler to connect the top left mark to the bottom right mark. Also connect the right top mark to the bottom left mark. Now you have something on your paper that looks roughly like an X. Drawing an X always creates four angles. Measure these angles carefully and record your measurements. The angles that are directly opposite each other are called vertical angles. Both of these angle pairs are called vertical angles, even though one of the pairs looks kind of horizontal. (If you haven't already, you'll eventually learn not to question how mathematicians name things.)

In Geogebra, place four points and use them to draw two intersecting (crossing) lines (see Geogebra Instructions). From the dropdown menu of the New Point menu, select Intersect Two Objects. Place a point at the intersection of the two lines. Click on the angle button and then on the points forming each angle, in a clockwise direction, to display angle measurements. Record the measurements of each angle.



Move your points around to create a different set of angles, and record their measurements. Repeat one more time so you have 3 sets of measurements.

Analysis

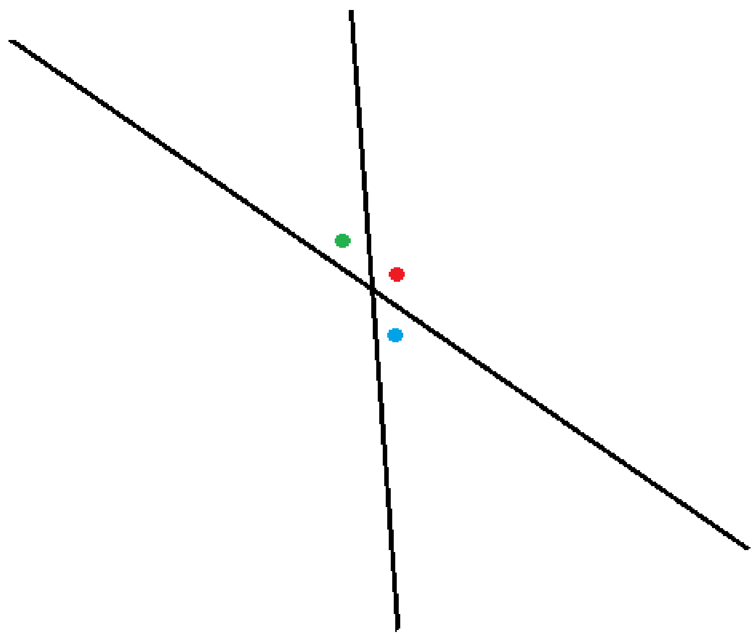
Question 1: What did you notice about the measures of vertical angles?

Question 2: Do you think that this is the same for any pair of vertical angles?

Question 3: Why do you think this happens? Whatever answer you want to put here is fine. Think about it for a while, maybe until tomorrow.

Vertical Angles, continued

Now let's have a look at Euclid Book 1, proposition 15.
http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI15.html



This picture shows the idea behind Euclid's proof. Two angles that are next to each other and add up to 180 degrees (a straight line) are called a *linear pair*. For example, the angle with the green dot and the angle with the red dot form a linear pair. Explain in your own words why the angle with the green dot is equal to the angle with the blue dot.

Alternate Interior ("Z") Angles

For this section, you need to know how to construct parallel lines. Parallel lines must have exactly equal distance between them, otherwise they will eventually intersect. To put it more precisely, at each point on one of the lines the minimum distance to the other line is exactly the same

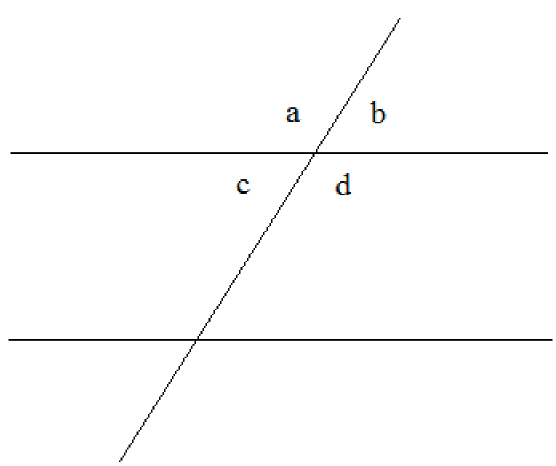
A ruler used alone is a very poor instrument for constructing parallel lines, unless you can conveniently use both the top and bottom edge of the ruler. Otherwise, you may think you are measuring out equal distances, but the minimum distance only occurs if you have a 90 degree angle between your ruler and the first line. You can use a protractor to make sure you have a right angle. You can also use a compass: http://www.youtube.com/watch?v=7pmonTIWunk.

Materials

Protractor
Compass
Paper

Procedure

Construct two parallel lines, and a line intersecting the first two, as shown in the picture. Do not copy the letters.

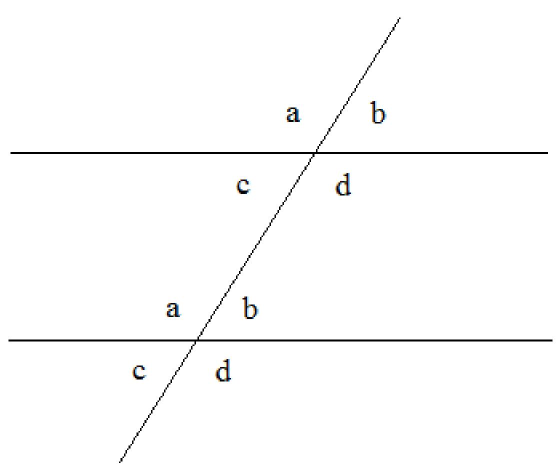


Consider angles a, b, c and d. From the previous experiment, you know that some of these angles are equal. In fact, there are only two different angles here. Label them x and y in your drawing. Notice that x + y = 180 degrees. Euclid says: If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles (Book1, Proposition 13). A more modern version of this says that these angles are *supplementary* because they add up to 180 degrees.

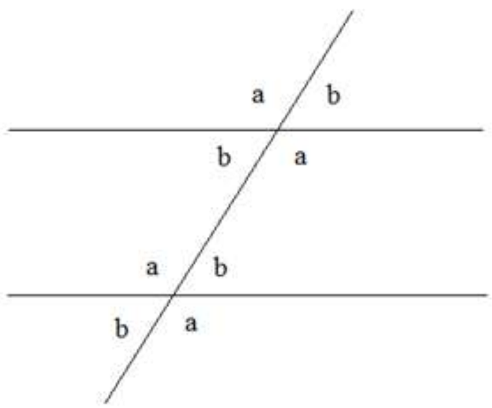
Look at the picture again. Only one intersection of lines has its angles labeled, but both are identical. Label every angle in your picture as either x or y. If necessary use your protractor to measure them.

Analysis

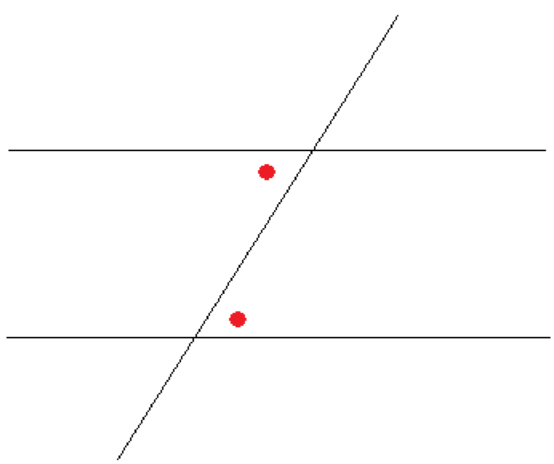
Because the two intersections are the same, we can start by labeling the corresponding angles. Surprisingly, these angles are actually officially called *corresponding angles*. Angle a at the top left corner of the top intersection and angle a in that same spot at the bottom intersection are a pair of corresponding angles. How many pairs of corresponding angles do you see? Corresponding angles are there because we have two parallel lines. If the lines are not parallel the angles will not correspond. Also, the other way around, if you draw your lines so that both angles labeled a are the same, you can be sure that your lines are parallel. This fact is sometimes used to construct two parallel lines.



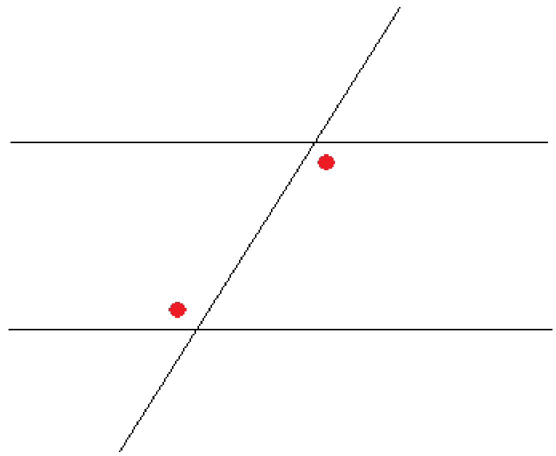
Next, we notice that many of the angles in the picture are identical because they are vertical angles.



The following angles are also the same:



and:

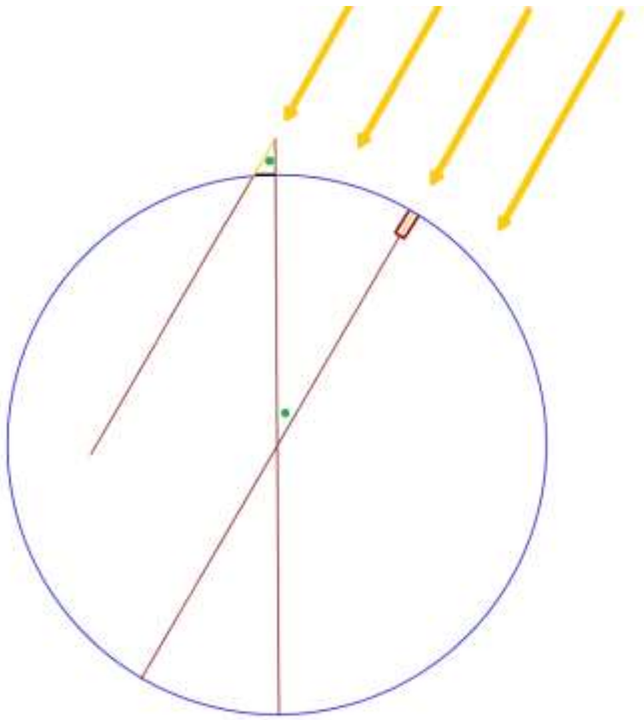


These angles are called **alternate interior angles**. They appear wherever you can draw a "Z" or the mirror image of a Z. *Remember that there are always two pairs of alternate interior angles.*

Euclid carefully proves that these angles are equal in Book 1, proposition 29. You should study this proof so you can learn how a good proof is written. To understand this proof you should also read Postulate 5 (referenced in the margin).

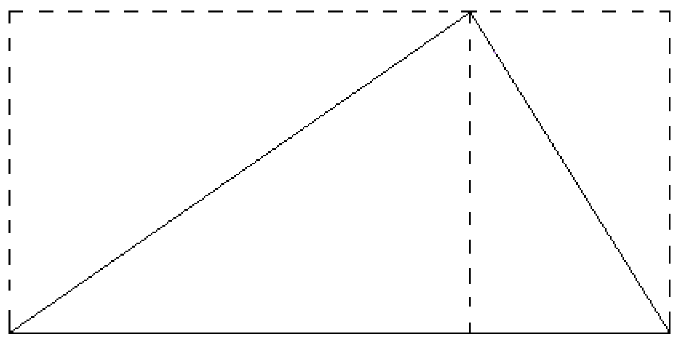
The *converse* of proposition 29 is that if the alternate interior angles are equal, the two lines must be parallel. The converse is basically a reversed "If...,then..." statement, and it is not always true. For example: "If there is a bad accident, the police are called" is true, but "If the police are called, there is a bad accident" is not a true statement since the police may be called for other reasons. Euclid proved that the converse of proposition 29 is always true: Book 1, proposition 27.

Eratosthenes measured the earth by using the angle between the sun's rays and a stick. He knew that the sun would shine directly down a particular well at a given time of the year, and he knew roughly how far away that well was. Ancient records tell us that he used 5000 stades for this distance, but we no longer know exactly how long a "stade" was. At his own location at that same time, he measured the angle in the triangle created by a stick and its shadow, as shown in the picture below, which is obviously not drawn to scale. He found that the angle was about 1/50th of a whole circle:



Calculate the circumference of the Earth in stades. An Olympic stade was probably 176 meters, which would give a result of 44,000,000 meters or 44,000 km for the circumference of the Earth. The actual value for the circumference at the equator is 40,075 km. Although our planet is not a perfect sphere as Eratosthenes assumed, his result is remarkably close considering how difficult it would have been to measure both the angle and the distance to the well accurately. Centuries later, Columbus would use a far less accurate estimate of the Earth's circumference to wrongly conclude that he had reached India instead of the Americas.

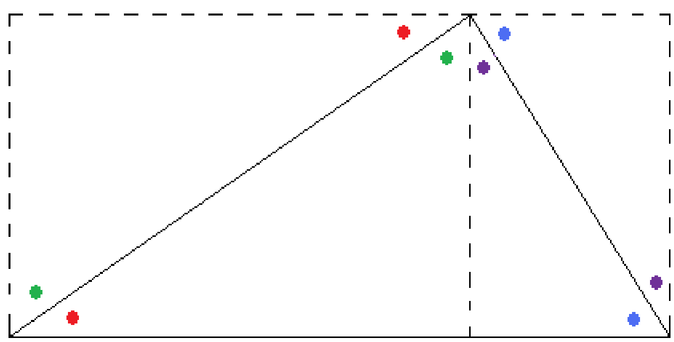
Now go back to your magic triangle in a box. Do you see any "Z" angles? Mark all that you can find with colored dots, using a different color for each pair. If you don't have colors, use shapes like a filled circle, an open circle, an asterisk, etc. How many pairs of Z angles did you find?



The Magic Triangle, continued

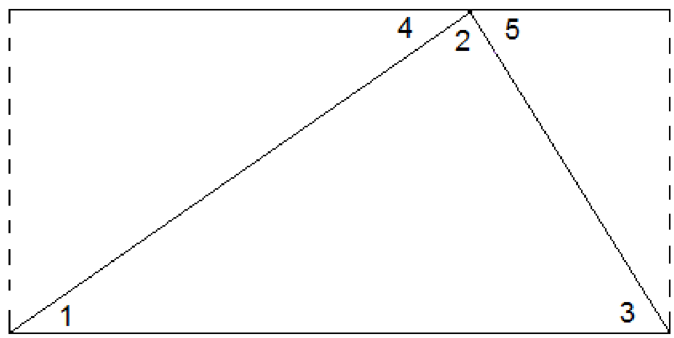
Now that you understand alternate interior angles and corresponding angles, you will be able to see exactly why the angles of a triangle always add up to 180 degrees.

In the following picture, each pair of Z angles is indicated by dots of the same color.



If you look carefully, you can see that the red and green angles must add up to 90 degrees, because they occupy a corner of a rectangle. The same is true for the purple and blue angles. Now add up all the angles in the triangle: (red + green) + (purple + blue) = 90 degrees + 90 degrees =180 degrees.

I created this proof to show how the sum of angles of a triangle is related to the more easily understood sum of angles of a rectangle. A more elegant proof can be constructed by omitting some of the lines:



Which angles in this picture are the same?

How does this picture prove that angles 1, 2 and 3 add up to 180 degrees?

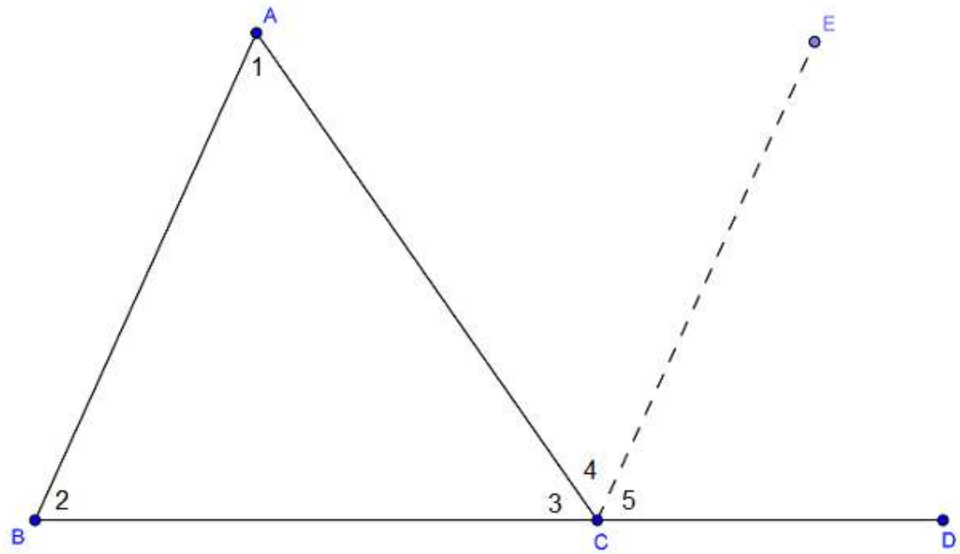
Euclid's proof is shown in the following experiment:

Materials

Geogebra
Paper - prefer construction paper
Ruler
Scissors

Procedure

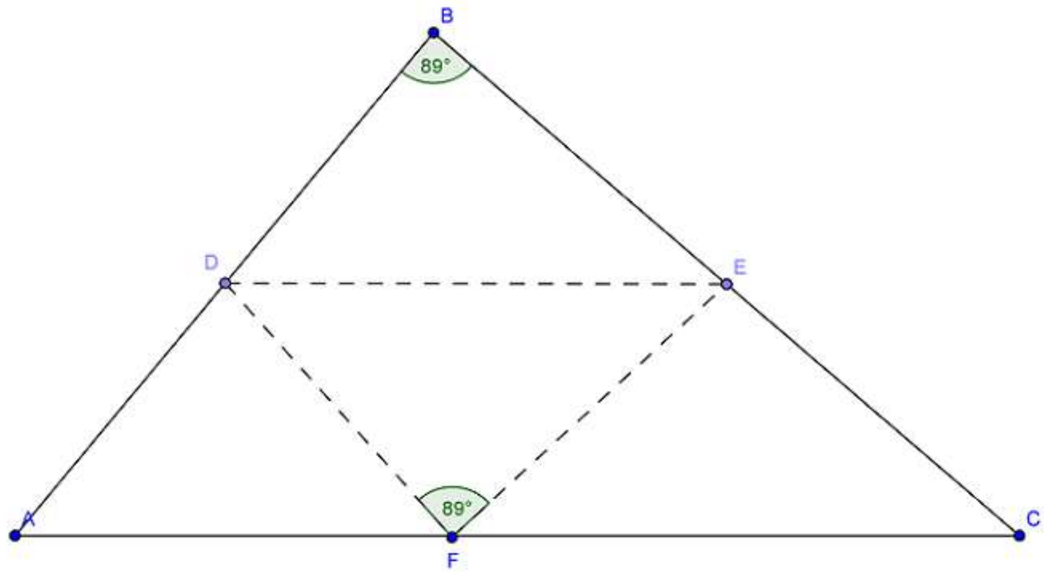
Create a triangle in Geogebra, similar to the triangle below. Construct a line parallel to segment AB through point C. Place point E on the line as shown, and create segment CE. Hide the parallel line so only the segment is visible. If you right-click on this line and select Properties, you can turn it into a dashed line if you want.



The angles in this picture are labeled 1 through 5. However, there are actually only 3 different angles here. Move the points of your drawing around so you can see which angles have the same measurements. Give reasons why these angles are the same.

Explain in your own words why the angles of triangle ABC must add up to 180 degrees. If you have difficulties, read Euclid Book I, Proposition 32.

Draw a large triangle on paper and cut it out. Label the angles 1, 2, and 3. There is no need to measure them. Turn the triangle over so that the numbers are on the other side. Take the vertex with the largest angle, which is B in the picture below, and fold it over so that it just touches the opposite side, making sure that the crease DE is exactly parallel to the bottom edge of the triangle. Fold the remaining two corners over so that they touch the same point. What do the angles add up to?



If you did your folding right, the corners of the triangle come together perfectly. Why is that? It looks like we still have a "magic" triangle. Actually, the vertices at points A and C fit at point F because our folding has created two isosceles triangles: ADF and FEC. What are isosceles triangles? Well, a hint would be that BE is equal to EC, and BE becomes EF when you fold it over. That gives FEC a certain symmetry so that corner C fits perfectly at point F when you fold it. Learn more about why point C fits just right in the next experiment, “What is an Isosceles Triangle?”

What is an Isosceles Triangle?

"Iso" is the Greek word for same, and isosceles literally means having the same legs. An isosceles triangle is a triangle that has two equal sides.

Materials

Protractor
Ruler
Paper
Geogebra
compass

Procedure

Draw a large triangle with two equal sides on paper. Carefully measure each angle inside the triangle using your protractor and record your measurements.

Draw a triangle using the polygon button in Geogebra (see Geogebra Instructions). Click on the angle button and then on the triangle to display angle measurements. Click on the small arrow at the bottom of the angle button and select "Distance or Length" from the menu. Click on each side of your triangle to display its length. Use the Move button to drag the points of the triangle around until you have two exactly equal sides. You may have to move more than one corner to get the sides exactly equal.

Record the measurements of each angle of the triangle. Use the move button to drag the numbers to a better spot if you have trouble reading them.

Move the points of your triangle around so you get a different triangle with two equal sides. Record the measurements of the angles for this triangle. Repeat this twice so that you have measurements for 4 triangles altogether.

Analysis

Which angles in your triangles are equal to each other?

To help you see a little better what is going on, take the triangle you drew on paper and turn it so that the unequal side is on the bottom. The two equal sides will be the top sides of the triangle. Divide the top angle in half by constructing an angle bisector as shown here: http://www.mathopenref.com/constbisectangle.html, and extend the line to the bottom of the triangle. Note that the word **bisect** means to cut into two (equal) parts. The root word *bi* means two, as in bicycle - a contraption with two wheels. The root word *sect* means to cut, as in dissection.

The original triangle has now been divided into two new triangles. If two triangles have the same angles and sides that are the same length we call them **congruent**. Are these two triangles congruent? If so, that would explain why the outside corner angles are the same. Well, they share a common side, which is the line you just drew. They also have another side which is the same because the original triangle had two sides of the same length. Because angle bisectors divide an angle into two exactly equal parts, the new triangles also have the same top angle.

When two triangles have two sides that are the same, and the angle between those sides is the same for each triangle, the two triangles are congruent. This is called Side-Angle-Side or SAS congruence. (The arrangement of the letters SAS shows that the angle must be between the two known sides.) You can see this yourself by using two line segments to draw an angle. Once you decide the angle and the length of the two sides, there is only one way to complete a triangle. (See also Euclid book 1, proposition 5.)

Cut out your paper triangle and fold it in half along the angle bisector. This should confirm that the two triangles are congruent.

Watch this video to see how you would use what you just learned about isosceles triangles to solve a geometry problem: http://www.youtube.com/watch?v=Tqh7RGWd480

Now that you have learned a lot about angles, watch this movie: http://www.youtube.com/watch?v=kqU_ymV581c

Write a simple conclusion about isosceles triangles and submit your lab report.

Angles of a Quadrilateral

We know that the angles of a rectangle are all 90 degrees, so the sum of the angles of a rectangle is 360 degrees. The same goes for a square. Squares and rectangles are quadrilaterals – shapes with four sides. There are many shapes with four sides that don't have right angles, so what about the sum of their angles? Is it still 360 degrees?

Materials

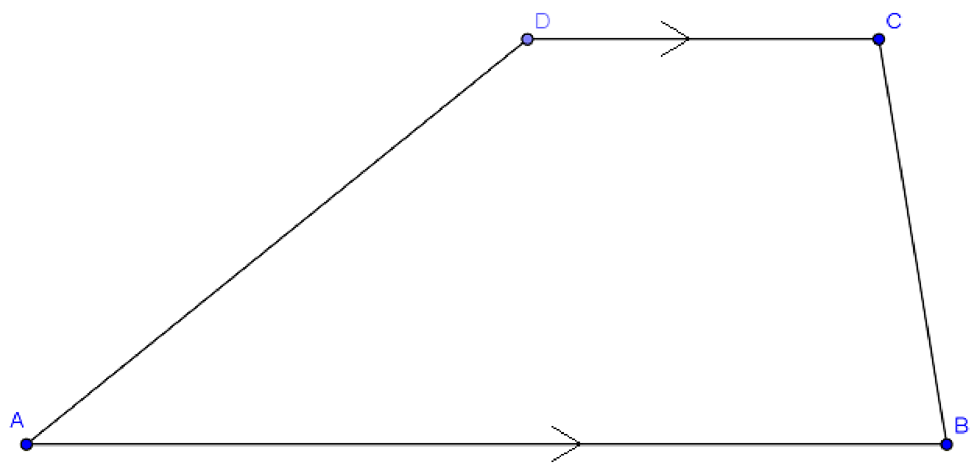
Ruler
Geogebra

Procedure

Use the Polygon button (button 5) in Geogebra to draw a shape with 4 sides. Use the Angle button (button 8) to mark the size of the angles. Add the angle measurements.

Move the points to change the shape of your quadrilateral. Add the angle measurements again. Repeat to get the angle sum of a third quadrilateral. Record your measurements and their sums.

Draw a random quadrilateral on paper. Draw a line from one corner to the opposite corner. This divides the quadrilateral into two triangles. What is the sum of the angles of the quadrilateral, based on the sum of the angles of the triangles?



This picture shows a special quadrilateral – a trapezoid. The arrows indicate that side AB is parallel to side DC. You can copy this in Geogebra by using Button 4, which has a parallel line in its options menu. Once you have side AB and point C you can construct a parallel line. Point D is a point on that line. If you right-click on the line and select "Show object" the line disappears, and you can replace it with a segment. Mark the size of the angles using Button 8. What is the sum of the measure of the angle at A and the angle at D? What is the sum of angle C and angle B? Why do these angles have a special sum? Could it have something to do with parallel lines?

Analysis

Once you feel comfortable with the idea that the sum of the angles of a triangle is 180 degrees, you can say that any quadrilateral is just two triangles.

For quadrilaterals with two parallel sides, some of the adjacent angles have a fixed sum. These angles are in between two parallel lines on the same side of the transversal (the line that crosses two parallel lines). These angles are called **same-side interior angles**. By extending the line segments that make up the sides you can show why the sum always has to be the same.

Angles of a Polygon - Interior

The word "poly" means many. "Polygon" means a shape with many angles. Triangles and rectangles are polygons. As we have seen, the sum of the angles of a triangle is 180 degrees. The sum of the angles of a rectangle is 4 times 90, or 360 degrees. Now we will consider shapes with more than 4 angles.

Materials

Pencil
Paper
Protractor
Geogebra

Procedure

Draw a shape with 5 angles using the polygon button in Geogebra (see Geogebra Instructions). This shape is called a *pentagon*. Click on the angle button and then on the shape to display angle measurements. Record the measurements of each angle of the polygon. Use the move button to drag the numbers to a better spot if you have trouble reading them.

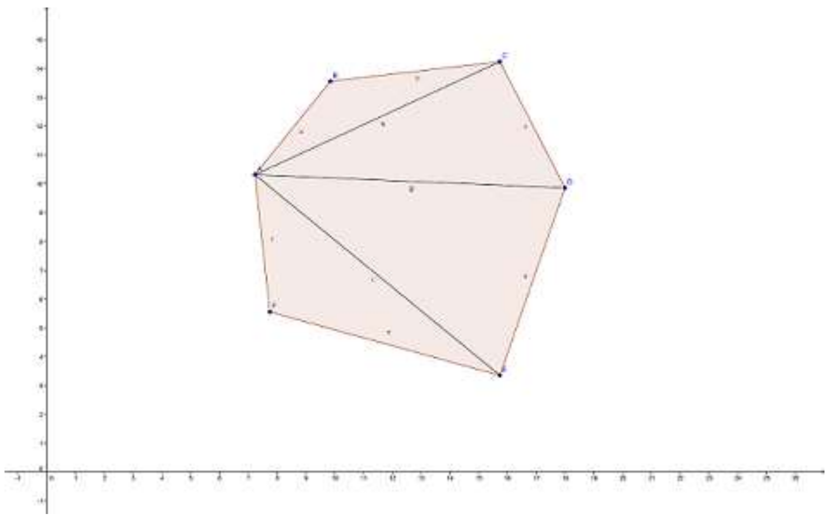
Draw four more polygons using Geogebra. You should vary the number of angles. Record the measurements of the angles. For each polygon you drew, record the number of angles and the sum of the measurements of those angles.

Verify that the sum of the angles of these polygons is given by the formula 180n - 360 (180 times n, minus 360), where n is the number of angles.

Analysis

Ooh, it's magic - NOT. By now you should know that there is probably a rational explanation.

Go back to your polygons and draw lines *from one vertex* (the point of one angle) to as many other vertices as you can, so that you divide each polygon up into triangles. How is the number of triangles related to the number of angles? Explain why the formula 180n - 360 gives the sum of the angles of a polygon.

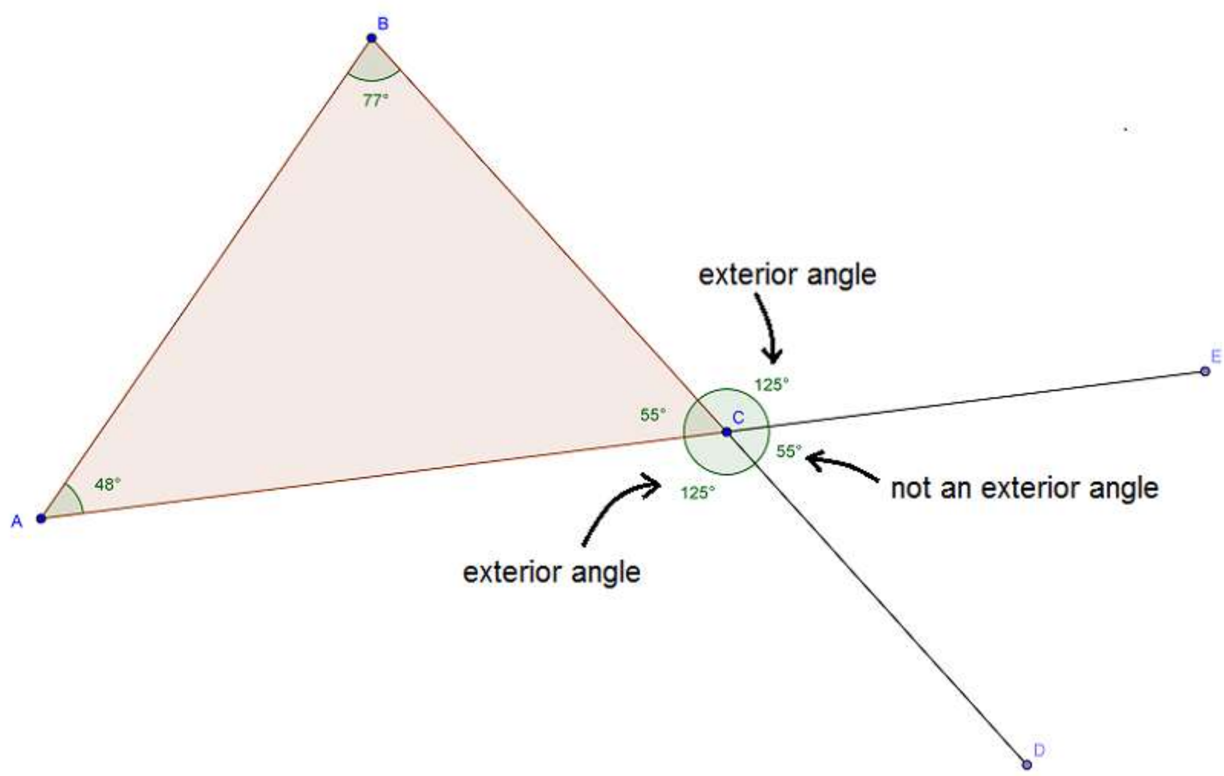


The sides of a *regular polygon* are all the same length. Draw a regular polygon in Geogebra, and mark its angles. What do you notice about the angles? Because we know the sum of the angles of a polygon, we can calculate the size of the individual angles of a regular polygon. How big are the angles of a regular pentagon? A triangle with sides of the same length is called an *equilateral triangle*. What is the size of the angles of an equilateral triangle?

If you know your algebra, you can solve the following problem: If a regular polygon has interior angles that are all 144 degrees, how many sides does it have?

Angles of a Polygon - Exterior

As a student, I found exterior angles confusing. There are two ways to draw an exterior angle but pictures always show just one. How would I know which one to pick? The answer is that it really doesn't matter. If you draw an exterior angle both ways, you can see that they are the same because they form vertical angles. There are two pairs of vertical angles. The two extended lines I have drawn also create a copy of the interior angle between them.



Materials

Geogebra

Procedure

Draw a triangle in Geogebra and mark the angles. At each vertex, draw one exterior angle by extending one of the sides. Record the measures of the interior and exterior angles with the points of the triangle three different positions.

Draw a square and its exterior angles. Record the measures of the interior and exterior angles.

Draw a polygon with 5 vertices. Record the measures of the interior and exterior angles.

For each shape, add the exterior angles and record the sum.

Analysis

Your measurements should show that the sum of the exterior angles of a polygon is always 360 degrees. This is nicely shown in an animation at http://www.mathsisfun.com/geometry/exterior-angles-polygons.html.

You can also see this by using simple addition and subtraction. We know that at each vertex an interior angle and its corresponding exterior angle always add up to 180 degrees (these angles are supplementary). That means that for a polygon with n angles, the sum of all the interior and all the exterior angles together is 180n degrees. Verify this fact for at least 3 different polygons.

Now we do some simple math: Fill in the blanks in the calculations below:

interior angles + exterior angles = 180n

interior angles = _______________ (from previous experiment)

interior angles + exterior angles - interior angles = 180n - ______________

exterior angles = ________

For some reason the fact that the exterior angles of a polygon add up to 360 degrees is a frequent subject of test questions. You should memorize it because chances are you'll need it at some point. It also makes it easier to solve the problem posted in the last experiment:

If a regular polygon has interior angles that are all 144 degrees, how many sides does it have?

Show how you would solve this problem in a different way, using your knowledge of exterior angles.

Read Euclid's proof about the exterior angle of a triangle: http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI16.html.

The Sides of a Triangle

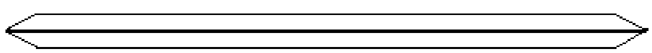
Materials

Paper
Construction Paper
Ruler
Scissors

1. The Triangle Inequality

Procedure

For this lab you will be drawing lines and cutting them out to work with. Use construction paper to get more sturdy lines. Your line cutouts should look like this, with the length of the line being the distance between the pointy ends:



Draw four lines with the following lengths: 2 inches, 3 inches, 4 inches, and 6 inches. Cut out your lines as shown. Now make as many different triangles as you can using these lines. For each successful attempt, record which three lines you used. Can you make a triangle with the lines 2 inches, 3 inches, and 6 inches? Why or why not?

Analysis

What properties do three lines need to have in order for you to be able to construct a triangle with them? This principle is called the triangle inequality.

2. SSS Congruence

Procedure

For this section of the lab you should have 1 or 2 people helping you. Create 3 copies of one of the sets of lines that you were able to make a triangle with. Each person should independently construct a triangle with these lines by placing them on a piece of paper. Carefully mark the points where the lines meet with a pencil, and remove the cut out lines. Use a ruler to connect the three points you marked on the paper. Measure the sides of your triangle to verify that they are still the correct length. Compare your triangle with those made by your assistants. This can be done easily by placing both sheets of paper on top of each other and holding them up to a window. If the triangles don't seem to match, try reversing one of the sheets so that you see a mirror image of the triangle when you hold the paper up to the light.

Analysis

Are all three triangles different, or the same? Are they the same if you flip one of the papers over? This principle is side-side-side congruence, usually called SSS congruence. Congruence means that both triangles have the same size and shape, although they may be mirror images of each other.

Angles and Sides - Which One Goes Where?

Materials

Ruler and Protractor, or Geogebra

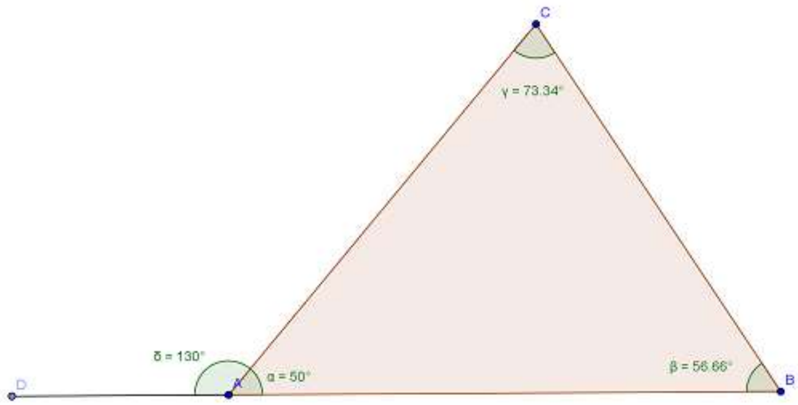
Procedure

Draw a triangle with three very different angles: a small angle (about 10 degrees), a medium angle (about 60 degrees), and a large angle (about 110 degrees). Measure the lengths of the sides. Where is the longest side in relation to the largest angle? Where is the shortest side in relation to the smallest angle?

Repeat this with another triangle that has different angles to see if you get the same result.

Analysis

When you stop to think about what you just discovered, you can see that it kind of makes sense. As an angle gets bigger, the side opposite it has to get larger to accommodate it. However, it is not so obvious when you go to prove it. Euclid's proofs make use of a property of isosceles triangles: the base angles of an isosceles triangle are the same. He also uses an *exterior angle*. Exterior angles are those that are created by extending the side of a figure. Either way you draw it, an exterior angle is always supplemental to the interior angle at that vertex (together they add up to 180 degrees). Euclid shows that an exterior (outside) angle of a triangle is equal to the sum of the two opposite angles, which means that the exterior angle is bigger than either one of the opposite angles by themselves.



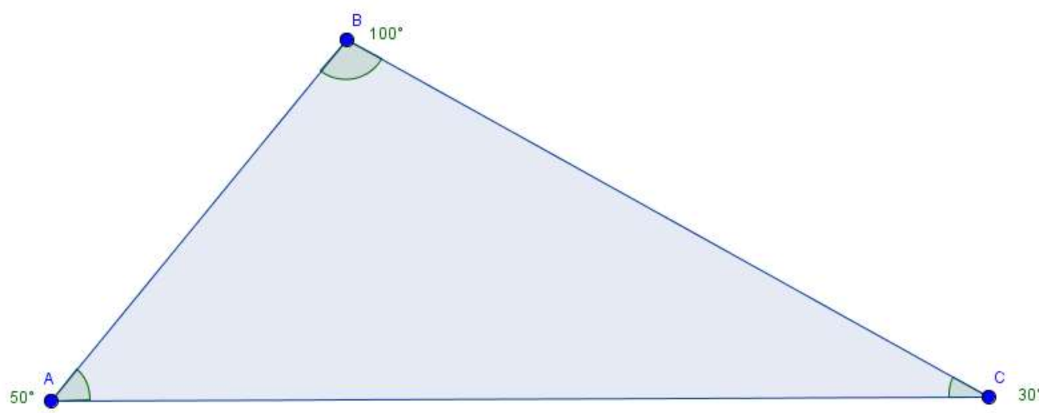
In this picture, the exterior angle at A is 130 degrees. The interior angle at this point is 50 degrees, which we would expect since these two angles form a straight line (they must add up to 180 degrees). The measures of the remaining interior angles are 73.34 and 56.66, so their sum is 130 degrees which is the measure of the exterior angle. Explain why this is true.

Read the following proofs:

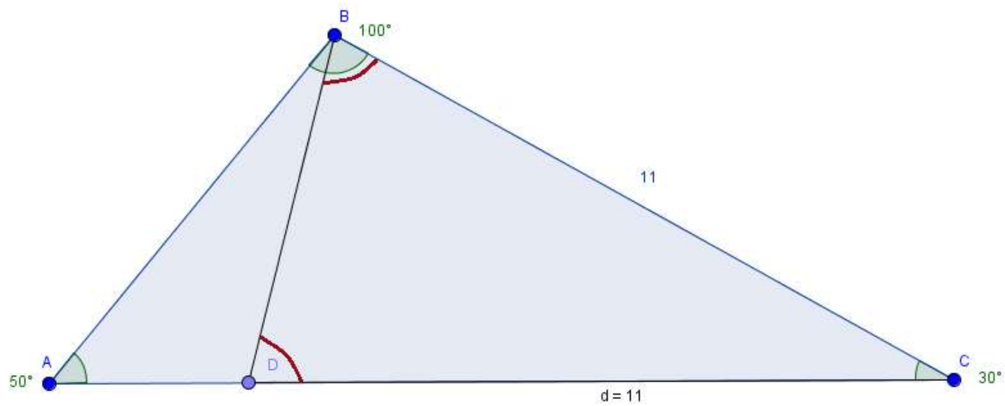
http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI18.html

http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI19.html

To illustrate these proofs, we will look at a triangle that very obviously has a longest side and a largest angle:

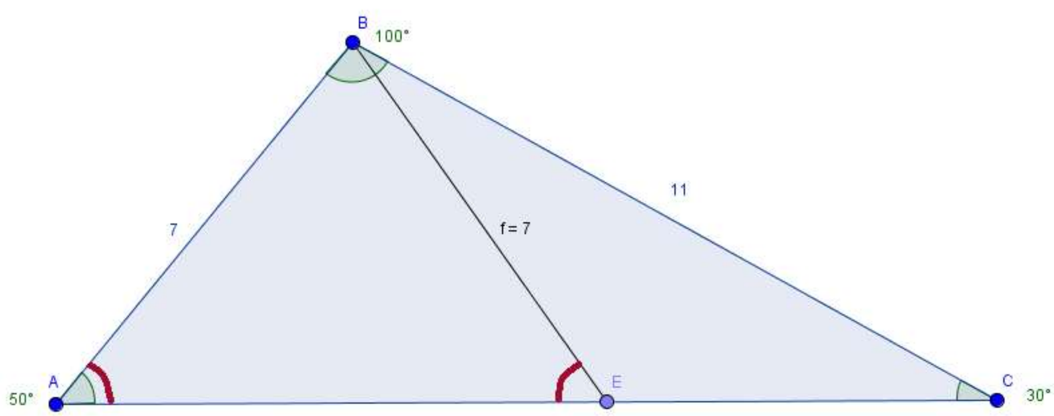


In this triangle, side AC is longer than the other sides. We want to prove that the angle opposite the longest side is always the largest.



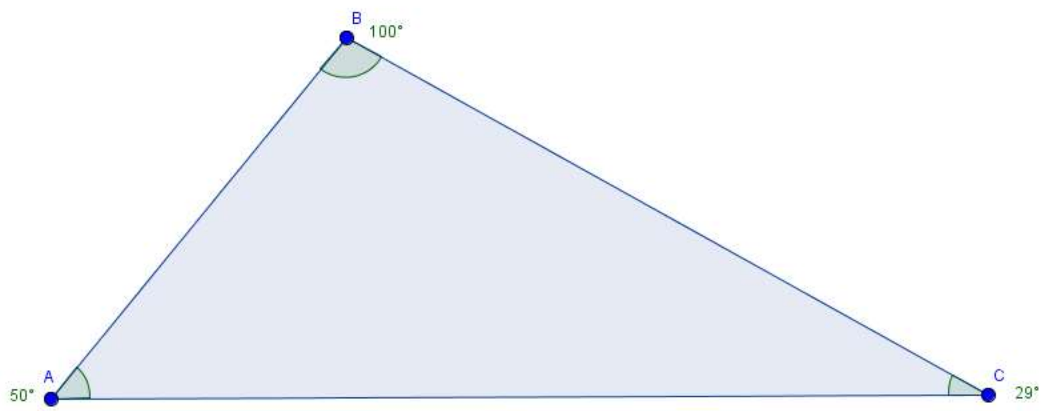
Since side AC, the longest side, is longer than side BC, we can place point D so that segment DC is equal to side BC, creating an isosceles triangle (triangle BDC). The angles labeled with red curved lines must be equal because they are the base angles of an isosceles triangle. The angle at D labeled with the red curved line is an exterior angle of triangle ADB, so it is definitely larger than angle A. (Remember that an exterior angle of a triangle is the same size as the sum of the opposite interior angles.) Angle B is the sum of DBC, which is bigger than angle A, and angle ABD. So, angle B is definitely bigger than angle A.

Since AC, the longest side, is also longer than AB, it is possible to place point E so that AE is equal to AB.



The angle at E labeled with the red curve is an exterior angle of triangle BEC. Again, this means that it is larger than the angle at C. Angle A, being the same size, is larger than angle C. Since we already saw that angle B is larger than angle A, A must be the largest angle in the triangle.

Once Euclid established that the largest angle in a triangle must be the one opposite the longest side, it was not so hard for him to prove that the longest side must be the one opposite the largest angle.



In this case we know that the angle at B is the largest, and we are trying to prove that side AC must therefore be the longest. We can say that if angle B is the largest, then angle A is not equal to it, so sides AC and BC are not the sides of an isosceles triangle. Otherwise the two base angles would be equal. So, side BC is definitely not equal to side AC. Based on the proof above, we can also say that if BC was the longest side of the triangle then angle A should be the largest because the largest angle is opposite the longest side. We can conclude that BC is not the longest side. The same reasoning goes for side AB: if it was equal to AC then angle C should be equal to angle B since the triangle would be isosceles. If AB was the longest side then angle C would be the largest, and it is not. AC is the longest side.

A Trick with Triangles: The Pythagorean Theorem

Pythagoras lived around 550 BC. He devoted most of his life to mathematics. He believed that reality is mathematical in nature, so if you want to know the truth about the universe you should study math. Many of his students lived with him in a community that we would probably consider to be a cult today. They followed strict rules of conduct, and kept mathematical discoveries secret.

Let's look at one of these discoveries, which fortunately is no longer a secret. We do not know if this particular discovery was made by Pythagoras himself, or by one of his students, since it was common in those days to attribute all big accomplishments to the master rather than the student. Even if you already know the Pythagorean Theorem, you can use the trick below to amuse your friends or slightly younger siblings.

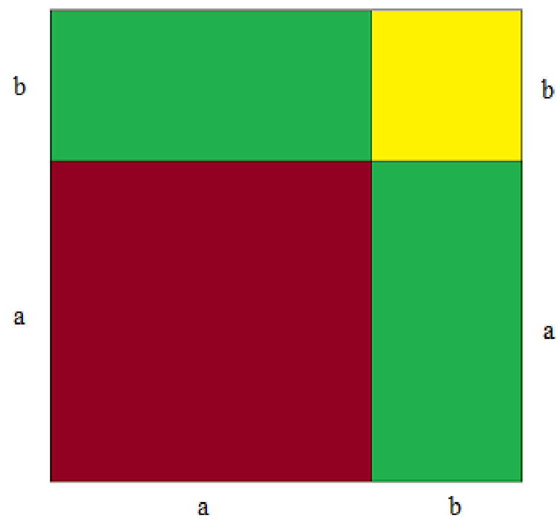
Materials

Ruler
Construction Paper - red, green, yellow, blue and white
Scissors
Geogebra

Procedure

First we are going to actually square (a + b) by creating a square with sides (a + b). To be able to see things clearly, we'll make 'a' significantly larger than 'b'. Let's make a 6 inches long, and b 3 inches.

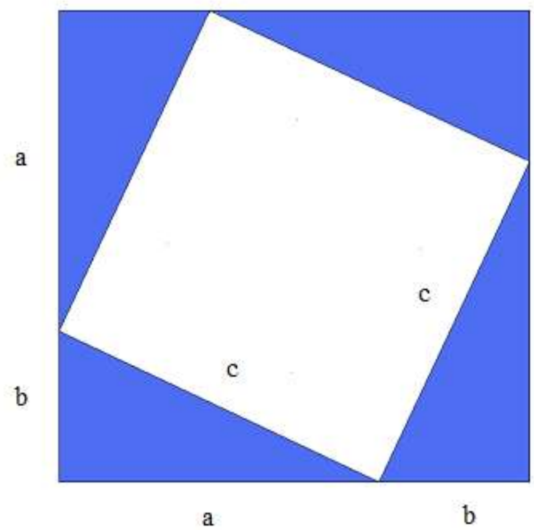
Use the colors shown below to make it easier to follow the directions.



The red square in the figure has sides a, so its area is a^2. The green rectangles have a longer side 'a' and a shorter side b. The area of each rectangle is ab. Since there are two of them, their total area is ab + ab, or 2ab. The little yellow square has sides b, so its area is b^2. You can see that this shows that $(a + b)^2 = a^2 + 2ab + b^2$.

Next, you need to use some blue construction paper and cut out a right triangle that has a (6") and b (3") as its two perpendicular sides. If you include the corner of your construction paper you will already have a nice right angle to start with. When you go to complete the triangle by drawing the last side, c, you'll see that you do not get a choice for the length of c. Pythagoras apparently drew quite a few of these triangles that contain one 90 degree [or "right"] angle, and he realized that once you pick the lengths for two of the sides, the length of the third side is already determined. We saw earlier that this holds true for any triangle - if you select the lengths of two sides and the size of the angle between them, there is only one way to complete the triangle (SAS congruence).

Because the Pythagoreans were so secretive, we cannot be sure how they eventually discovered the formula that gives the relationship between the sides of a right triangle. The best we can do is to try to re-create what they might have done. They were interested in harmony and perfection, so if they were studying something with a 90 degree angle they might have gathered four of these things to have a perfect 360 degrees. Create 3 more triangles that are identical to the first one you made. Arrange them as shown in the picture to create a square with sides (a + b).

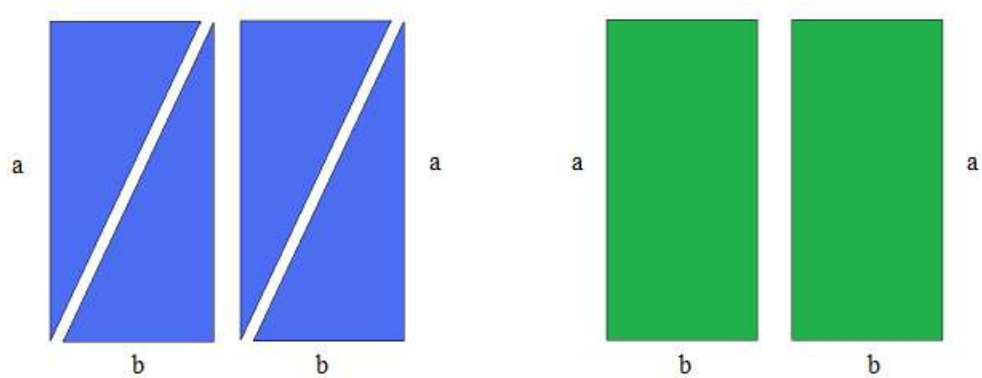


Now consider the following questions:

1. You now have two squares with sides (a + b). Place them next to each other so you can see the sides have the same length.

2. Consider the empty space inside the square you made with the triangles. What is its area? Use white construction paper to make a square that fits in the empty space.

3. Take apart the shape you made with the triangles. Put the triangles together in pairs to make two rectangles. What is the total area of these two rectangles?



4. Put the triangle shape back together, with the white square in the empty space. Now you have a big square. What is left if you remove a total area of 2ab from this square?

5. Look at the other square that is made out of squares and rectangles. Remove a total area of 2ab from this square also. What is left?

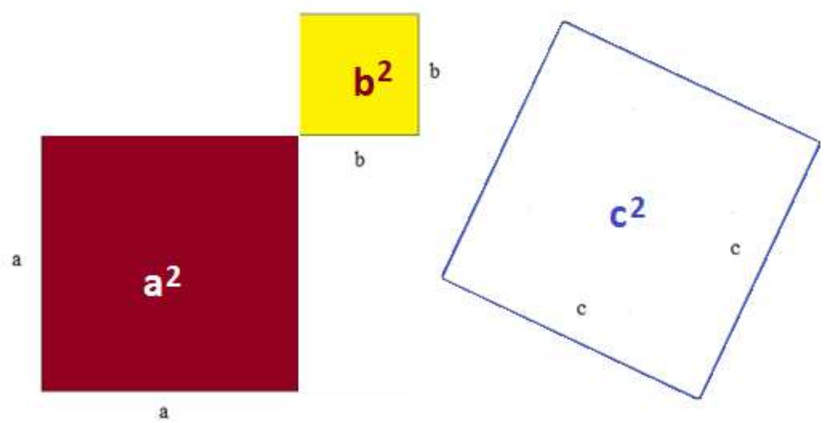
6. Both squares had the same area to start with. You removed 2ab from the first square, and 2ab from the second square.

7. The big square with the squares and rectangles now has a red square and a yellow square left. The triangle shape only has a white square left.

8. Create a formula from the remaining squares.

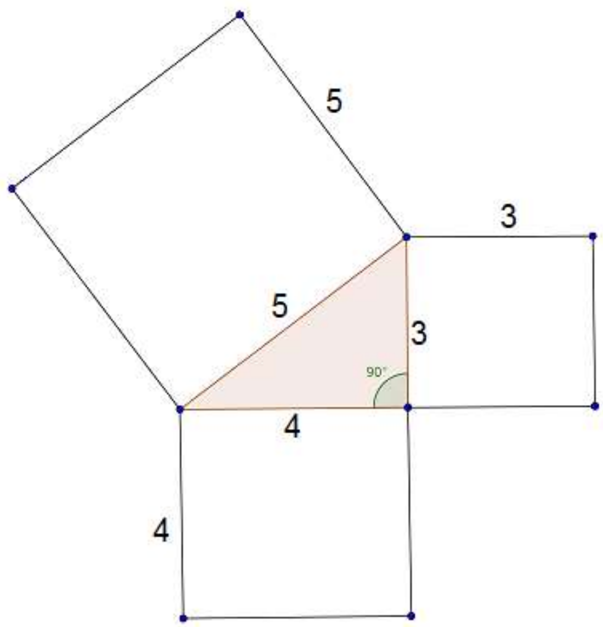
Analysis

The area of the red square is a^2, and the area of the yellow square is b^2. Together these two squares must equal the white square, which has an area of c^2. We started with two large squares that both had an area of $(a + b)^2$. We subtracted equal amounts from both squares, so the remaining amounts must be equal.



Here is my favorite explanation of the Pythagorean Theorem:
http://www.youtube.com/watch?v=b6vzK2ULRTs

Note that the Pythagorean Theorem can only be used for right triangles, which are triangles that have one 90 degree angle.



If you select 3 for a, and 4 for b, then $a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25 = c^2$. Now c = 5, which is a nice integer result. The numbers 3, 4, and 5 are a *Pythagorean triple*. Because it is the simplest and most obvious Pythagorean triple you will likely not see it on important tests. However, you can take a triangle with 3, 4 and 5 centimeter sides and make each side twice as big. Does this triangle still have a right angle? Try it out in Geogebra, because this shows an important point about proportions.

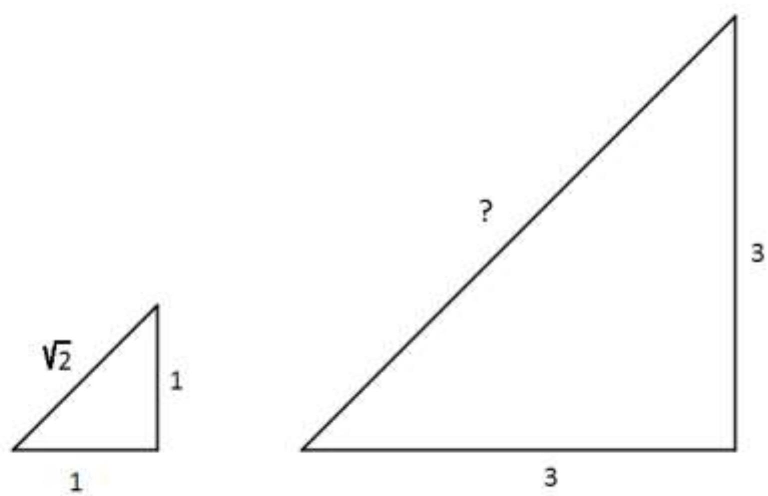
There are infinitely many Pythagorean triples. Another common one is 5, 12, and 13.

We can say that if a triangle has a right angle, then $a^2 + b^2 = c^2$. The converse of this would be that if $a^2 + b^2 = c^2$, then the triangle must have a right angle. Do you think that this is also true? Use Geogebra to check if you are right.

Start with a right triangle. Now make side c (the hypotenuse) a little longer. What happens to the right angle? What do you have to do to make the right angle smaller?

Special Triangles

The Pythagorean Theorem can help you figure out the proportions of two important special triangles. The first triangle is created by drawing a square with sides of 1 unit, and then drawing a diagonal. This creates two triangles. Because we made them from a square, these triangles are isosceles. Draw one such triangle and figure out its angles. This type of triangle is named for its angles, so you should remember them. The Pythagorean Theorem tells you that when the two shorter sides are 1 unit, the hypotenuse is $\sqrt{2}$. As you saw earlier, all proportions remain the same when we multiply all the sides by the same number. If you know that these proportions are 1, 1, $\sqrt{2}$, then you can enlarge this type of triangle as needed:

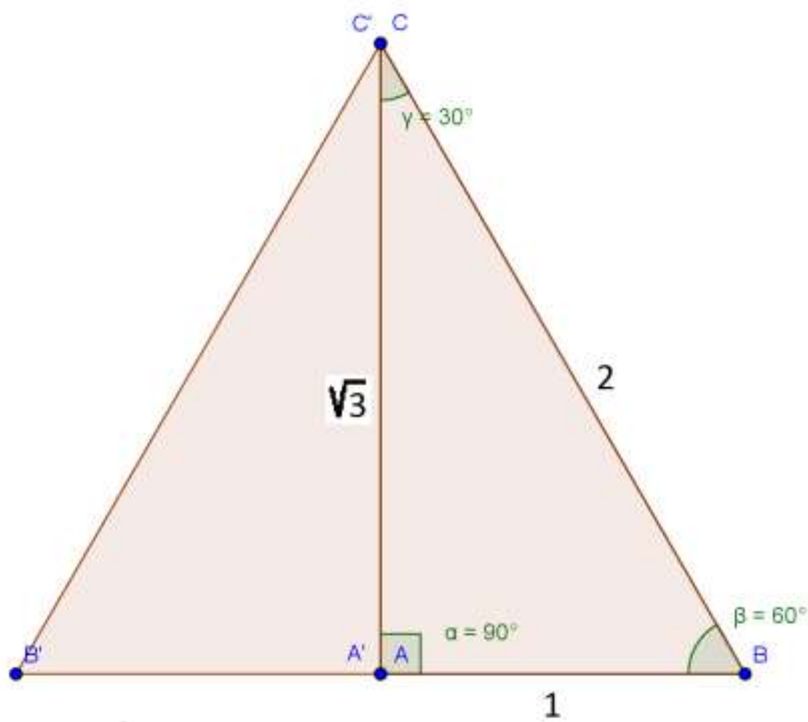


Because the two sides of the larger triangle are three times longer than those of the smaller triangle, the hypotenuse will also be three times longer. The hypotenuse of the larger triangle is 3 times $\sqrt{2}$, or $3\sqrt{2}$.

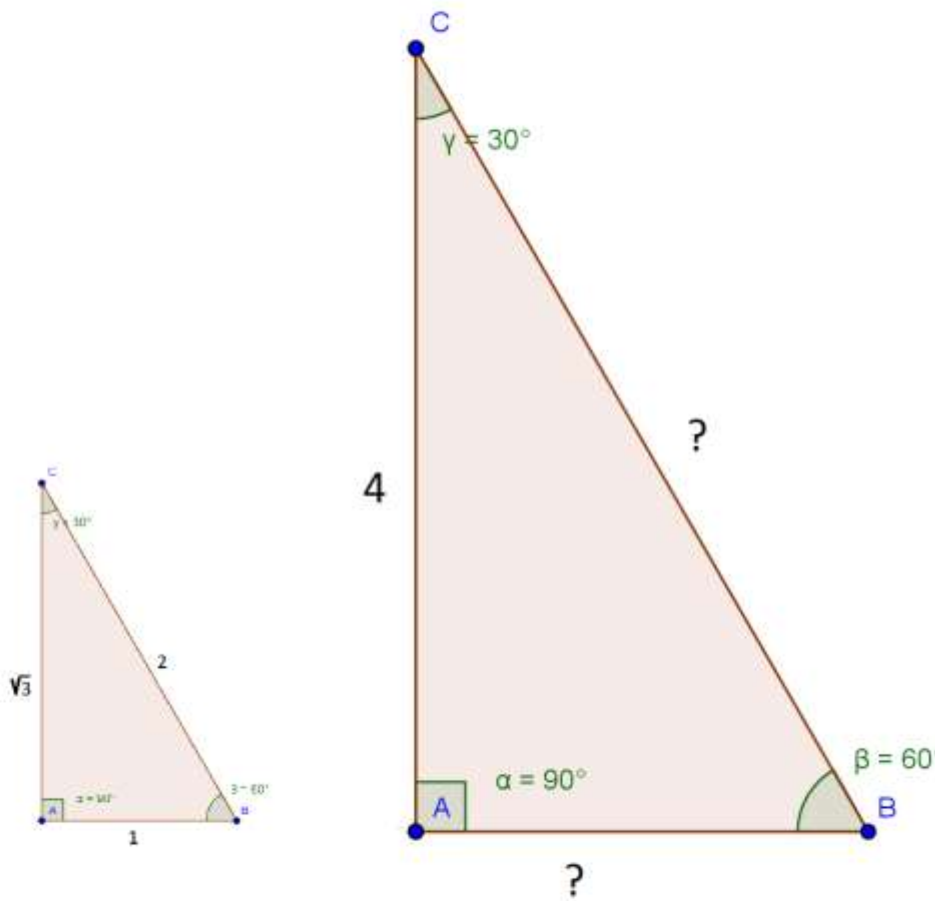
Our second special triangle is a 30 - 60 - 90 triangle. Set Geogebra to show 0 decimal places (Options -> Rounding), and use the Polygon button to draw a triangle with 30, 60 and 90 degree angles. This is not really easy to do, so remember to use the information from the experiment "Angles and Sides - Which One Goes Where" to help you. Next, select the 9th button from the left and choose "Reflect object about line". Select the triangle, and then the second-longest side of the triangle (the side opposite the 60 degree angle). A second triangle appears that is the mirror image of the first. Together, the two triangles form a larger triangle. All of the angles of this triangle are 60 degrees. If all the angles are the same, are all of the sides equal?

If we say that these sides are 2 units long, then by the Pythagorean Theorem our original 30 - 60 - 90 triangle must have sides that are 1, 2 and $\sqrt{3}$ units. Do your own calculations to confirm these numbers. This is the second important proportion to remember.

If you have a 30 - 60 - 90 triangle where the shortest side is 10 units, you know that the sides must be 10, 20 and $10\sqrt{3}$ units long.



If needed, you can both shrink and expand your 30 – 60 – 90 triangle. The next image shows a large triangle with a side of 4 units that corresponds to the side that is $\sqrt{3}$ in the smaller triangle. To get the other side lengths, you can first shrink the original triangle so that the side that is $\sqrt{3}$ units long becomes 1 unit long. This is easy to do: just divide all of the sides by $\sqrt{3}$. So, if you start with side lengths 1, $\sqrt{3}$, 2 and divide by $\sqrt{3}$, you end up with sides that are $\frac{1}{\sqrt{3}}$, 1, $\frac{2}{\sqrt{3}}$. Next, multiply everything by 4 to make the middle side 4 units long. The new side lengths are $\frac{4}{\sqrt{3}}$, 4, and $\frac{8}{\sqrt{3}}$.



Those are some awkward looking fractions, but remember that a fraction is really a division. $\frac{4}{\sqrt{3}}$ means $4 \div \sqrt{3}$. Just plug that into a calculator to get an answer of about 2.31. Then the longest side will be twice as long as that, or about 4.62. Alternatively, your teacher may want you to put the fraction into a different form. Teachers don't like square roots on the bottom of fractions. You can fix that by multiplying both the numerator and the denominator by the square root, like this:

$\frac{4}{\sqrt{3}}$ x $\frac{\sqrt{3}}{\sqrt{3}}$

Here you are really multiplying by 1, so that doesn't change the value of the fraction. The square root of 3 multiplied by itself is 3, just like $\sqrt{25}$ x $\sqrt{25}$ = 25. $\frac{4}{\sqrt{3}}$ x $\frac{\sqrt{3}}{\sqrt{3}}$ = $\frac{4\sqrt{3}}{3}$.

Practice:

1. If a 30 – 60 – 90 triangle has a hypotenuse of 5 inches, what are the lengths of the other sides?

2. If a 30 – 60 – 90 triangle has a middle side of 5 inches, what are the lengths of the other sides?

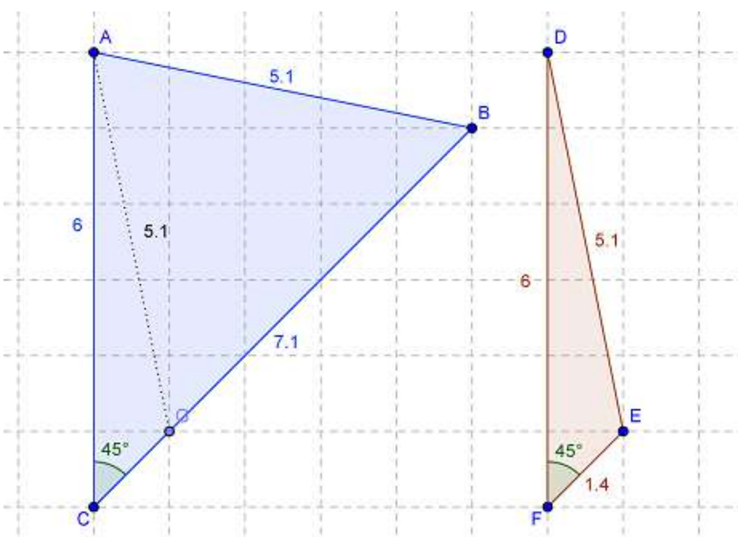
@$$ Congruence?

So far, we have seen three ways to tell if two triangles are the same when there is limited information:

1. SSS Congruence - all sides are the same
2. SAS Congruence - two sides and the angle between them are known to be the same
3. ASA or AAS (SAA) Congruence - all angles and a side are the same

In each case, when you draw the three known parts of the triangle, its shape is already determined.

If you look carefully at the letter combinations, you will see that one possibility is missing. There is no ASS congruence, which is lucky for teachers who would want to keep reminding students to write that as SSA instead.



Here is a picture of two triangles that have two sides the same, and one angle. Notice that the known angle is not between the two sides, because if it was it would not be possible to create two different triangles. The dotted line inside the blue triangle shows how the two triangles are related to each other.

This took me a while to do, and I used the grid view (right-click on the drawing surface and select Grid) to help get at least one of the sides the same. Create your own, different picture in Geogebra to show that there are many different possible triangle pairs that satisfy the ASS condition. Note that you can copy my picture and then drag the points around. You can also move an entire triangle to a different spot. Right-click on a triangle and select Object Properties to change the color. Do you think it is possible to find a pair of such triangles where the known angle is obtuse?

For a right triangle, if we know the length of one of the sides and the length of the hypotenuse, the known angle (90 degrees) is not between the two known sides. However, thanks to the Pythagorean Theorem we already know the length of the remaining side, so the triangle is fully determined. Two right triangles are congruent if the hypotenuse and one of the sides of one triangle are the same length as those of the other triangle. This is called the Hypotenuse-Leg Theorem rather than “ASS congruence of right triangles”.

Next Up: Test 1

If you look at the schedule you can see that there is a test in the next topic. The purpose of this test is to check on what you have learned; not to have you search for the answers in the course material. Creating a study sheet is a good way to prepare for a test. Go through all the topics you have finished and write down any terms or conclusions that you might not recall on a test. Do not include things that you are sure you already know well. Do write down things that you think you know but could possibly get confused about while taking a test. Submit a copy of your study sheet in the same document as your lab report for this week’s experiment.

Where is the Middle?

Geometry tests may require you to find the midpoint of a line segment in a coordinate system. There is a somewhat complicated-looking formula that goes with that, but you can discover it yourself in this experiment.

If you are not sure about coordinates, watch the movie here: http://www.youtube.com/watch?v=HdrCwFNcXGU&feature=related.

You can practice using coordinates in Geogebra. Right-click on the drawing surface and select Grid. Dashed grid lines appear. The horizontal numbered axis on your drawing surface is called the x-axis, and the vertical axis is the y-axis. Notice how the positive and negative numbers are placed on the axes.

Use button 2 to place a point at any intersection of grid lines. The coordinates of the point appear on the left side of the screen in the Algebra section. If this section is not visible you can go to View → Algebra to make it appear. Move the point around and notice what happens to the coordinates. Which of the two numbers indicates the distance along the x-axis? This is called the x-coordinate. The other number is called the y-coordinate. Move your point so that both coordinates are negative. In this quarter section of the drawing surface, below the x-axis and to the left of the y-axis, all points have negative coordinates.

Materials

Ruler
Graph paper or Geogebra

Procedure

Draw a coordinate system on your graph paper. If you don't have graph paper you can work in Geogebra instead (select View -> Grid from the top menu). Mark the point (0,0) and the point (6,8), and draw a line segment between these points. [1]Where do you think the middle of the line segment is located? Measure to see if your guess was right.

Draw a line segment from (0,0) to (6,0). [2]What are the coordinates of the midpoint? [3]Could you figure that out without drawing the segment?

Draw a line segment from (2,0) to (8,0). [4]What are the coordinates of the midpoint of this segment? Use the Pythagorean Theorem to confirm your answer by calculating the distance between each endpoint and your proposed midpoint.

Let's try some vertical line segments. [5]Where is the midpoint of the line segment between the points (0,0) and (0,8)? [6]How about for the segment between (0,3) and (0,11)?

[7]Use a ruler (or create a midpoint in Geogebra) to find the midpoint of the line segment that runs from point A at (3,5) to point B at (9,7).

Analysis

To find the middle of the line segment from (2,0) to (8,0), you need a number that is between 2 and 8. That doesn't take too long with trial and error, but guessing at it could be a problem when the numbers are large. If you have taken two tests in a course and you want your current test grade, you would take the average of the two numbers. That gets you a number that is exactly in the middle between the two grades. In the same way you can take the average x-coordinate: $(2 + 8) \div 2 = 5$. You can also use the average y-coordinate to find the midpoint of the segment between (0,3) and (0,11), which is (0,7). What is more interesting is that you can take both the average x-coordinate and the average y-coordinate to find the exact midpoint of any line segment. The actual formula is:

Midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, where (x_1,y_1) and (x_2,y_2) are the coordinates of the two endpoints.

Why does that work? Well, coordinates work with straight lines at a 90 degree angle. When you specify two coordinate points, you can draw a nice right triangle to go with them. In the picture below, E is the midpoint of segment AD, and F is the midpoint of BD. The small triangles AEC and CFB both have a 90 degree angle. Because we are working in a coordinate system, we can see easily that CF is equal to AE, and BF is equal to CE. By the Pythagorean Theorem, if right triangles have two equal sides then the third sides must also be equal. AC equals CB, so C is the midpoint of AB.

[8]Try to find the midpoint of a segment that has negative coordinates in its endpoints. You also need to be able to work backwards to find one of the endpoints if you only know the midpoint and the other endpoint. [9]If the midpoint of segment AB is at (4, -5) and the coordinates of point A are (-19, -19), what are the coordinates of point B?

Are Triangles With the Same Angles the Same?

Materials

Paper
Protractor
Centimeter Ruler
Geogebra

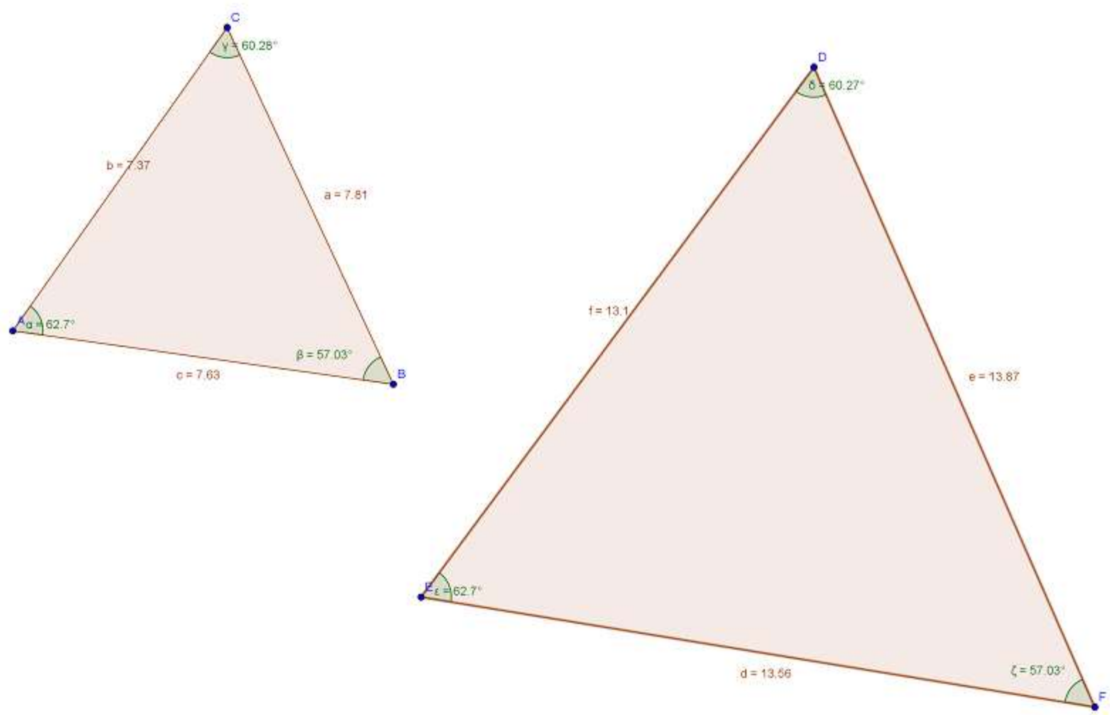
Procedure

For this part of the lab, you must work very accurately and carefully. Measure all angles to the nearest degree and all sides to the nearest millimeter. Draw a large triangle that fills most of your paper. Measure the angles and the sides. Next, draw a smaller triangle *that has the same angles.*

Measure and record the lengths of the sides of the larger triangle. Order the sides from largest to smallest. Do the same for the smaller triangle: record the length of its sides and order these lengths from largest to smallest. Now we will calculate the ratio of these sides. Divide the length of one of the larger triangle's sides by the length of the *corresponding* side of the smaller triangle. Do the same for the other two sides. What do you notice about this ratio? What happens if you divide the length of the smaller triangle's sides by the length of the corresponding sides of the larger triangle?

Next, take two of the sides of the larger triangle, and find the ratio by dividing one by the other. It doesn't matter if you divide the larger of the two lengths by the smaller, or the other way around. Then look for the corresponding two sides on the smaller triangle. Divide the length of one of the sides by the length of the other, *in the same order* that you used for the larger triangle. Show your calculations. The two ratios should be the same.

In geometry, changing the size of an object while keeping its proportions the same is called a **dilation** (even if you are actually making the object smaller). Create a triangle and measure the angles using button 8. Then place a point at another spot and select Dilate Object from Point by Factor from the menu on button 9. Follow the instructions that appear when you hover your mouse over the button to dilate your triangle by a factor of 0.3. A new, smaller triangle appears. Measure the angles of this small triangle. Are they the same as the angles of the larger triangle? What do you notice about the shapes of the two triangles as you move the vertices around with the move button? We say that these triangles are *similar*. This is also known as AAA (Angle-Angle-Angle) similarity.



If you draw two triangles with the same angles, and one pair of corresponding sides of these triangles are the same, will all sides be the same?

Analysis

When we know that the angles of two triangles are the same, *and* one of the sides of one triangle is the same length as the corresponding side of the other triangle, we can say that the two triangles are congruent (they are the same size and shape, although one may be a mirror-image of the other). This idea is formally known as Angle-Side-Angle (ASA) Congruence or Angle-Angle-Side (AAS) or SAA Congruence depending on whether the side you are looking at is between the two given angles or not. ASA and AAS or SAA are really all the same thing since knowing two angles means knowing the third also. The different order of the letters just indicates the way in which the information is provided in a given problem or stated in a certain proof. Just remember that it really means that you know all the angles (so there is AAA similarity) and a side as well, which pins it down specifically.

We can use the ratio of sides to calculate the length of an unknown side of a triangle. For example, consider two triangles whose angles are the same size. One triangle has sides of length 3cm, 4.5 cm and 6 cm. The other has sides of length 4.5 cm and 6.75 cm, but the length of its longest side is unknown. What is the length of the missing side?

Read more about this topic at http://library.thinkquest.org/20991/geo/spoly.html

How to Shrink a Triangle

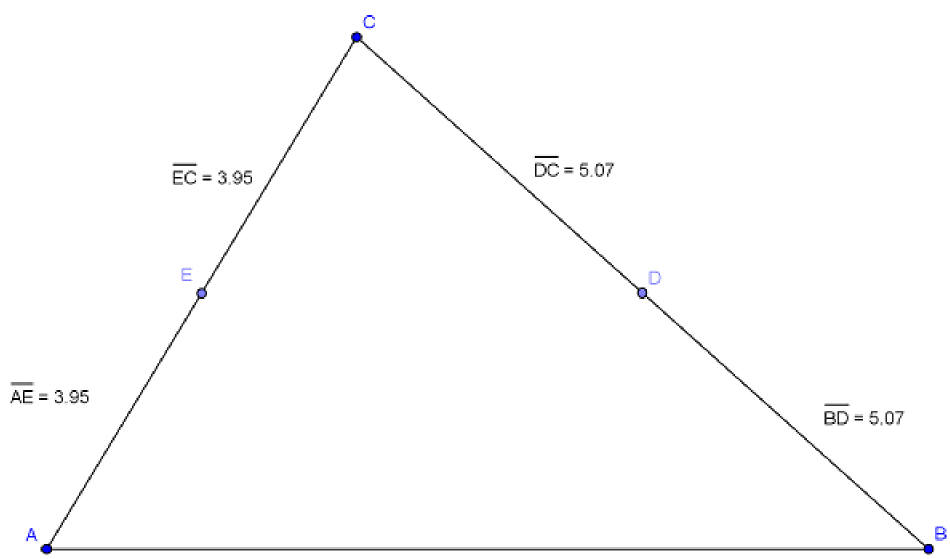
Just wash it in very hot water. Alternatively, if you have a paper triangle, follow the procedure below.

Materials

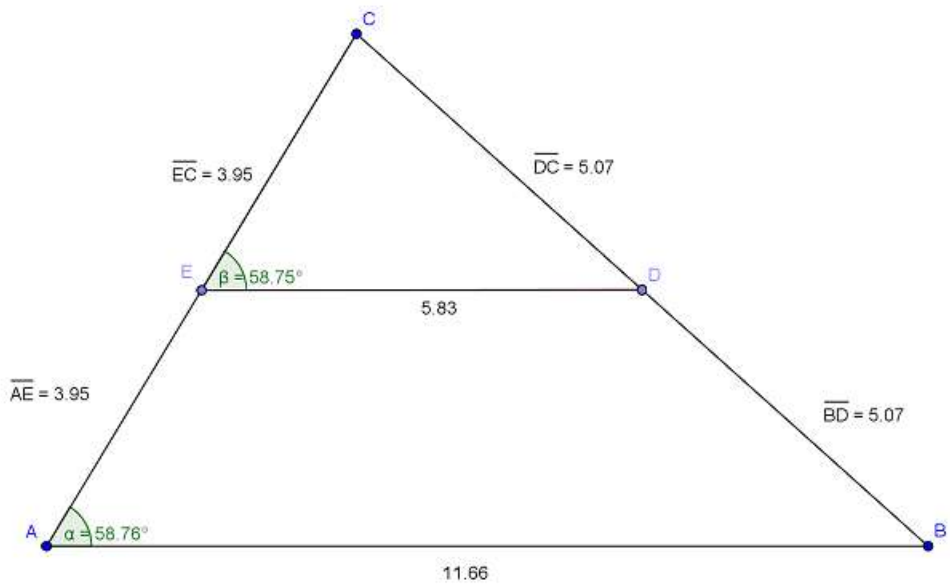
Triangle
Ruler
Protractor
Geogebra

Procedure

Start with a large triangle drawn on paper. Carefully mark the midpoint of two of the sides of your triangle (points D and E shown below).



Then draw a line segment between the midpoints:



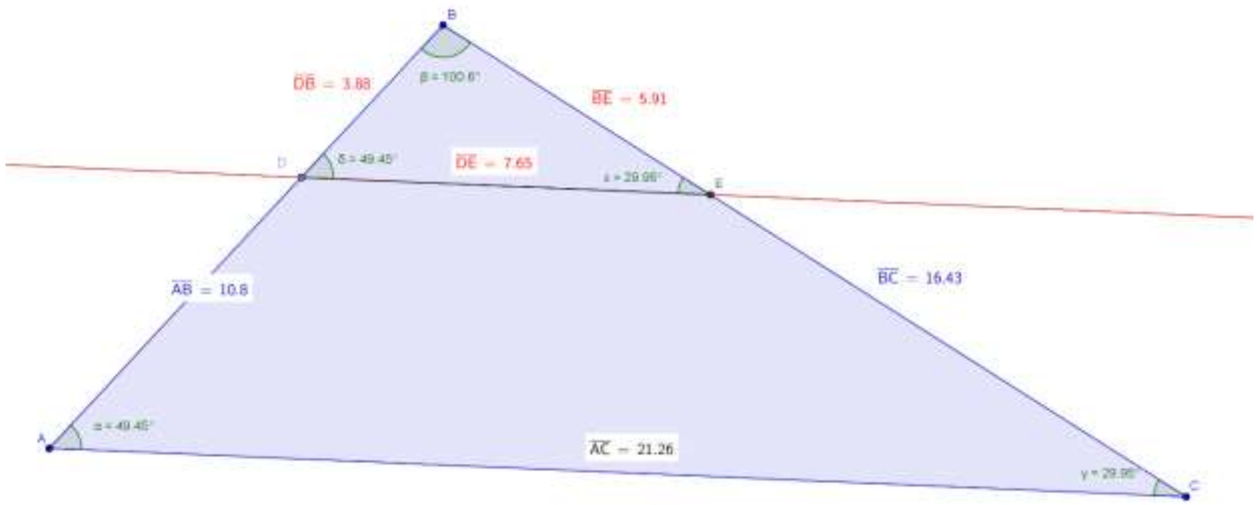
Measure the line segment you just drew. Notice that it is exactly half the size of the bottom of your triangle, just like in the picture above. (If it is not you may want to re-measure your midpoints).

There is now a smaller triangle inside your original triangle. Carefully measure the angles of both the original and the smaller triangle. Are they the same?

Do you think that line segment DE is parallel to AB? How can you tell by looking at the angles?

Note that you can also shrink a triangle by, for example, a factor of 3. Just mark two points on the sides that are 1/3 of the way from the top instead of 1/2. However, test questions on this subject always use the midpoints.

Now draw a similar picture using Geogebra. Create triangle ABC using the polygon button. Then place points D and E on two of the sides, and create segment DE. Mark all the angles with their sizes. Move points D and E until the angles of the smaller triangle are the same as those of the larger triangle. If you then create a line through point D parallel to the third side by using button 4, you can see that segment DE is parallel to the bottom of the large triangle.



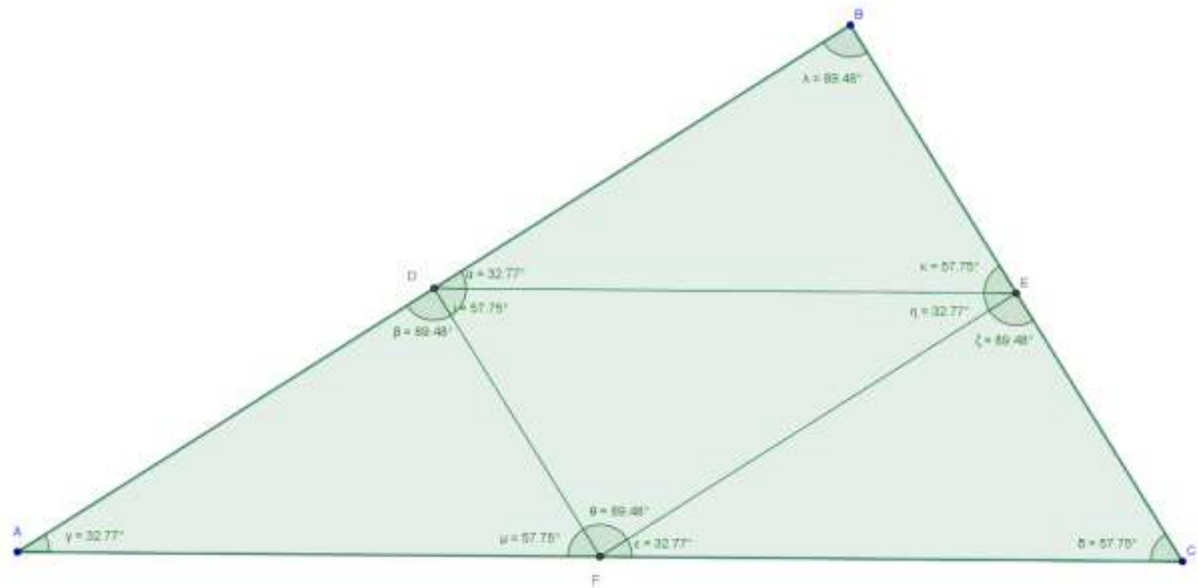
Check the ratio of the sides of the smaller triangle to the corresponding sides of the larger triangle. Why is this ratio constant?

Analysis

This experiment shows that a line segment drawn between the midpoints of the sides of a triangle is one-half the length of the bottom side, and parallel to it. This interesting fact is a popular subject for test questions, so you should take note of it.

The fact that a line segment drawn between the midpoints of the sides of a triangle is one-half the length of the bottom side, and parallel to it, is caused by the fact that we have shrunk our original triangle so that it is now half the size. You may be surprised at how easy it is to shrink a triangle this way. Recall from a previous lab (What is an Isosceles Triangle') that when you draw two sides of a triangle, the third side is already determined. This is SAS (side-angle-side) congruence. Also, as you saw in the last experiment, when two triangles have the same angles their sides are proportional. The converse of that is that when the sides of a triangle are proportional they have the same angles, which is true. When you shrink two of the sides of a triangle by the same factor, and leave the top angle the same, you create a similar triangle that has the same angles as the original. This principle is called **SAS similarity**.

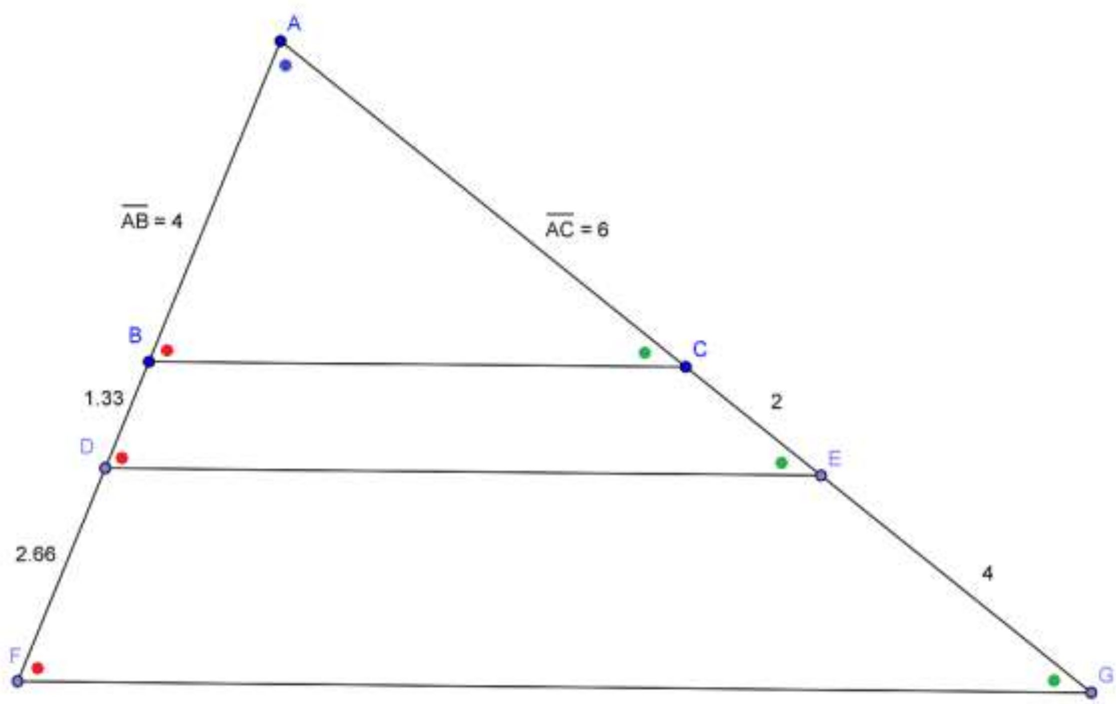
On paper, create a new triangle and connect the midpoint of the sides to create a smaller triangle inside it. The larger triangle and the smaller triangle share their top angles. Next, turn your triangle so that a different angle is at the top, and again create a smaller triangle by connecting the midpoints of the sides. Repeat again for the last angle. Now there should be four smaller triangles inside your larger triangle. Are all of these triangles congruent? Well, there are now a lot of parallel lines in your drawing. Use your knowledge of corresponding and alternate interior angles to check if the small triangles all have the same set of angles. Then check the lengths of at least 1 set of corresponding sides.



The smaller triangles were created using the midpoints of the sides. You can see which angles are the same, and you can prove that they should be. Write a proof showing that triangle EFD is similar to triangle ABC.

More Similarity: Splitting Your Sides

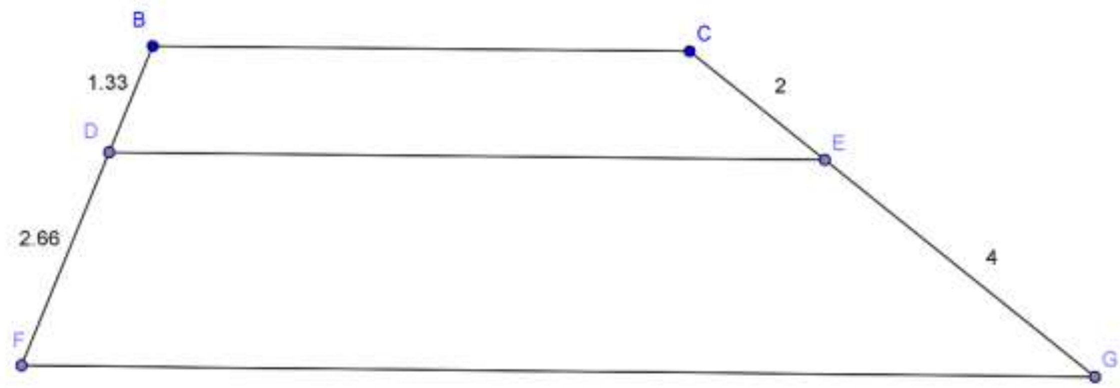
As we saw before, we can shrink a triangle and create a similar triangle. In the same way we could extend two of the sides of a triangle to create a larger triangle that is similar to the one we started with. For example, I could draw a triangle and then extend two of the sides by 1/3 of their original length (creating triangle ADE as shown below). Because the top angle stays the same and the two sides are in perfect proportion, the angles of the original and the extended triangle are identical. The triangles are similar by SAS similarity, and the base of the new triangle is parallel to the base of the original triangle, since the corresponding angles are the same. I can then extend the sides some more, say by 2/3 of their original length, and create a third triangle with exactly the same angles. The base of this triangle is again parallel to the base of the other two triangles.



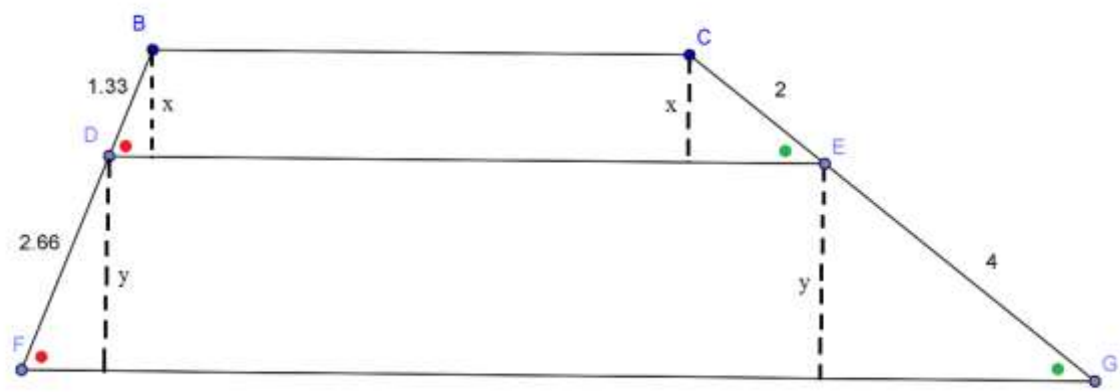
If the line segments BC, DE and FG are parallel, the angles with the red dots (and the angles with the green dots) are equal, because they are corresponding angles. The converse is also true: If the corresponding angles are equal the lines must be parallel. When you try this out in Geogebra you really only need one line that crosses two parallel lines, and you only need to measure two angles.

If one of the measurements along the sides was missing, you would be able to find it by looking at the other measurements. For example, if the measurement for DF was not given, we could look at the ratio of CE to EG and say that it must be equal to the ratio of BD to DF. CE/EG = BD/DF, or .5 = 1.33/DF, and so DF = 1.33/.5 = 2.66.

Sometimes a question may present this picture with the top part cut off, like this



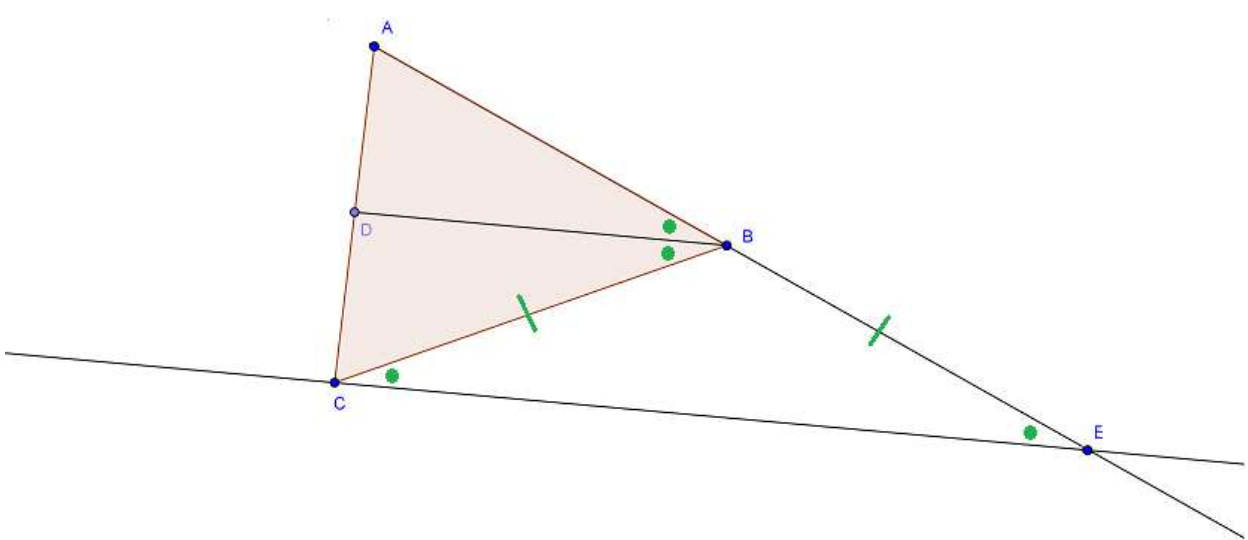
The information supplied is usually that the three horizontal line segments are parallel, and you are then again asked to find one of the side measurements like EG. You may not recognize such a picture as part of similar triangles, but you can still show that the side segments must be proportional. Just drop some perpendicular lines to create two pairs of similar triangles:



The triangles with the red dots have two equal angles created by the intersection of a line with two parallel lines, and they both have a 90 degree angle. If two angles are the same, all three angles are the same, so the triangles are similar. One pair of sides is in the ratio x to y, so all of their corresponding sides are in the ratio x to y. The triangles with the green dots are also similar, and again all their corresponding sides are in the ratio x to y. If we need to find the length of EG, we would say that the ratio x/y is the same as BD/DF or 1.33/2.66. This is equal to CE/EG, and again we find that EG = 4.

All of this proportionality is summarized by the **Side Splitter Theorem**: If a line parallel to one side of a triangle intersects the other two sides, then it divides the sides proportionately. The corollary to this is that if three parallel lines intersect two transversals, then the intercepted segments are proportional.

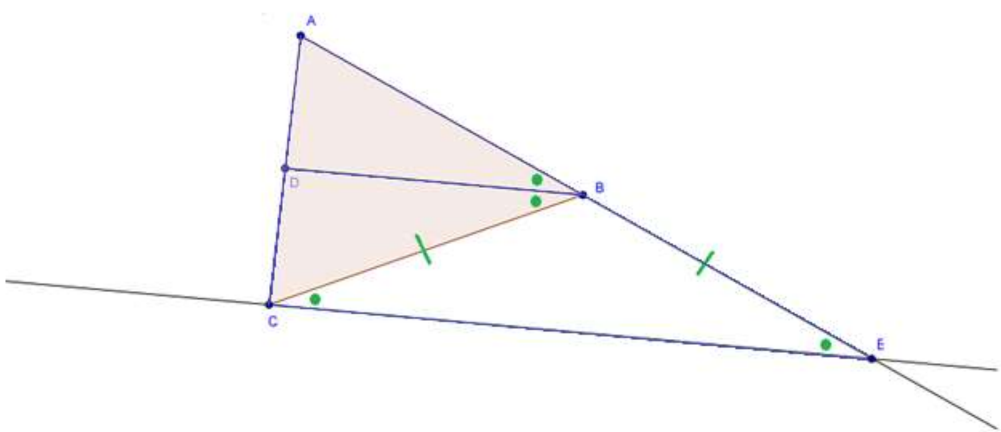
Another variation of the amazing shrinking and expanding triangle is found in Euclid, Book VI proposition 3. It goes like this: Take any triangle and bisect the top angle. Draw the angle bisector so that it touches the base, dividing the base into two sections. These two sections are now in the same ratio as the remaining sides of the triangle are to each other. To show you Euclid's explanation, I have bisected the top angle of triangle ABC, and then turned it sideways so that it would correspond better to the images above:



A line parallel to BD has been drawn through C. The Side Splitter Theorem tells us that for the largest triangle in the picture, ACE, line BD divides the sides proportionately. The ratio of AB to BE is the same as the ratio of AD to DC.

Euclid explains why the ratio AB/BC is also equal to AD/DC. The two angles with the green dot at B are equal because we bisected B. The green dot at C can be placed because it is one of two alternate interior angles. The green dot at E can be placed because line AE intersects the two parallel lines, forming equal corresponding angles. Now we can see that triangle CBE is isosceles (because the base angles are equal), so we can place two green marks to show which segments are equal. From the explanation above you know that AD/DC = AB/BE. However, since BE = BC, we can also say that AD/DC = AB/BC. This is the **Triangle-Angle Bisector Theorem**. This theorem says that if a ray bisects an angle of a triangle, then it divides the opposite side into two segments that are proportional to the other two sides.

Now look at the picture again. Triangle ADB is similar to triangle ACE:



The Triangle-Angle Bisector Theorem is really just a variation of the Side Splitter Theorem.

Trapezoids: Chopping a Triangle

Materials

Triangle
Ruler
Protractor
Calculator
Scissors
Optional: Geogebra

Procedure

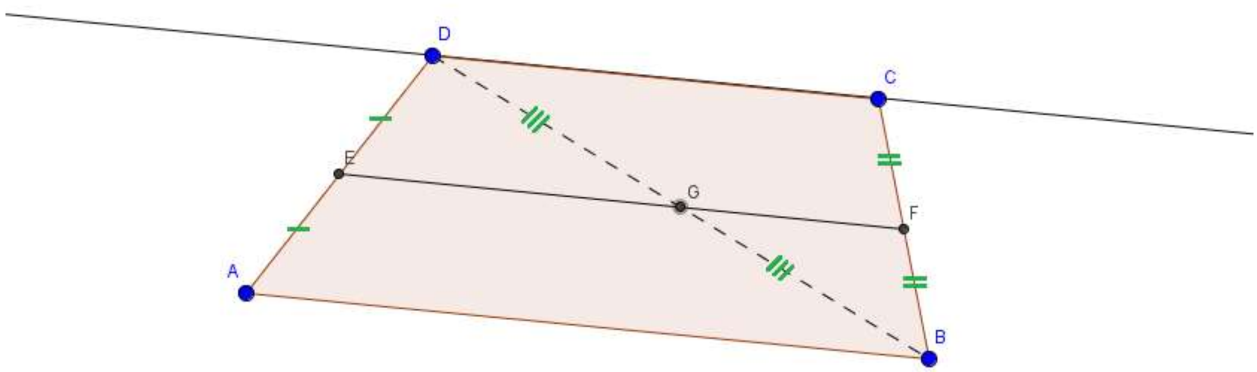
Starting with a random triangle, measure two of the sides. Divide the measure of each side by 3, and mark the thirds. Draw two lines to divide the triangle into 3 parts. Both lines should be parallel to the third side of the triangle, which we will call the bottom side. Cut along the top line to remove the top part of the triangle. The remaining shape is a trapezoid.

Your trapezoid will have a smaller top and a larger bottom side, called the bases, that are parallel to each other. The remaining two sides are not parallel, and may or may not be equal in length. Measure and record the lengths of the top and bottom sides. There is still a line drawn on your trapezoid. This line runs from the middle of one side to the middle of the other side. Measure and record the length of this line. This length should be the average of the length of the two bases. Add the lengths of the bases and divide by 2 to get the average. Is this average equal to the length of the middle line?

Analysis

It doesn't seem particularly surprising that a line drawn through the middle of a trapezoid should be average in length between the two bases, and it illustrates an important idea that will come in handy when you take a physics course. In physics, if the speed of an object is changing at a constant rate, we can simply use the average speed to determine how far it travels. Starting at the bottom, the width of the trapezoid decreases at a constant rate, determined by the slopes of the sides, until we get to the top. In the middle of the trapezoid, the width is exactly in between that of the starting point and the endpoint. That works the same for a triangle, which also decreases in width at a steady rate as we move from the bottom to the top. Half way, we are exactly in between the starting value (the length of the base of the triangle), and zero (the width at the top). The width in the middle is the average of the bottom and the top, which is one-half of the bottom. We already saw that this is true in the experiment "How to Shrink a Triangle". So, just like in the physics problem, can we simply use the average width of the trapezoid to determine the area (width times height)? Draw a trapezoid in Geogebra and determine its area. Show your calculations. Then use the dropdown menu on the angle button to mark the area.

Geometrically, we can see that the line segment connecting the midpoints must be the average of the two base lengths. We can draw one of the diagonals of the trapezoid:



To create this trapezoid, I constructed a line parallel to AB through the point D. Then I adjusted point C so that CD would be parallel to AB. The diagonal line divides the trapezoid into two triangles. If we place the point G so that it is the midpoint of the diagonal, then EG will be the midsegment of triangle ABD. As we saw previously, EG will be half the length of AB, and parallel to it. By the same reasoning, GF will be half the length of DC and parallel to it. Because DC is also parallel to AB, GF is parallel to AB. Because EG and GF are both parallel to AB, these two segments form a straight line segment, EF The length of EF is halfway between the length of AB and the length of CD.

Tangled Triangles

For this topic you need to know how to solve proportions by cross-multiplying. The fact that cross-multiplying works at all is kind of amazing – just try it out with some random fractions. For example, we know that $\frac{3}{4} = \frac{9}{12}$. You can always multiply the top number of one fraction by the bottom of the other, and vice versa, to get two equal results: 3 times 12 = 36, and 4 times 9 = 36. This means that if one of the values is unknown, you can easily find it by this method. For $\frac{2}{3} = \frac{10}{?}$, we know that 3 times 10 has the same answer as 2 times ?, so the missing value must be 15.

Oddly enough, cross-multiplying even works if there are two missing values, but the catch is that they both have to be the same:

$$\frac{4}{x} = \frac{x}{6.25}$$

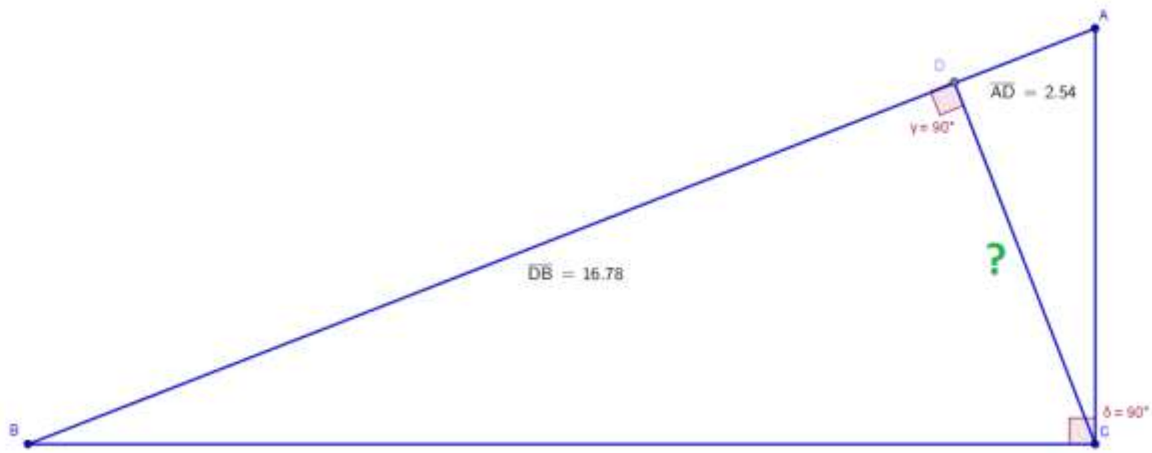
Here x represents an unknown length. We know that x times x (which is x^2) has the same value as 4 times 6.25, and that value is 25. If $x^2 = 25$, we can guess that x is 5 in this simple case. When the numbers are more complicated, we have to take the square root: $\sqrt{25} = 5$.

Materials

Geogebra

Procedure

Draw your own version of the following picture in Geogebra:



All you need is a large *right* triangle ABC, and a point D on the hypotenuse. I made sure my large triangle had a right angle by using the grid (right-click on the drawing surface while the "Move Graphics View" button is active, and select "Grid"). The lengths of the sides should be clearly unequal, so that you can see the proportions more easily. Measure angle BDC and move point D until the angle is as close to 90 degrees as possible. Mark the lengths of segments BD and AD. Your lengths may be different from the ones shown in the picture.

The object of this exercise is to find the length of segment CD without measuring it. At first that may seem impossible. However, if you look at the triangles closely you will see that all three triangles in your picture (the small, medium, and largest triangle) appear to have the same proportions. Of course, "appear to have" is not sufficient in geometry; you have to be able to prove that. To do so, you must show that all of these triangles have the same angles.

Look at the right angle of the largest triangle. It has been divided into two parts by the segment with the question mark. Although we do not know the size of each of these smaller angles, we do know that they add up to 90 degrees. Label these angles a and b. Next, use your knowledge of triangles to label the remaining unknown angles in the picture. You should not need letters other than a and b, but make sure you can justify why you place each letter.

Once you label all the angles, you should see that the triangles are definitely similar.

Solving the Problem

Tangled triangles look confusing, but it helps to note that each triangle has a hypotenuse (the "slanty" side, which is the longest), and two sides adjacent to the right angle. In problems like this, these last two sides are usually clearly unequal, so you can think of them as the short side and the long side. If you have difficulty solving the problem, you may want to draw all three triangles separately. Each triangle has a hypotenuse, a short side, and a long side. Carefully label them with the measurements provided. Mark the length you are trying to find with an x.

Now use proportions to find the length of segment CD. Show your calculations. Measure the length using Geogebra – it should be very close.

Analysis

The fact that all of the angles of a triangle add up to 180 degrees allows us to sort out these tangled triangles. Once we prove that the angles are the same for each triangle, we can create equal proportions. Right triangles have two sides and a hypotenuse. One of the sides has been drawn longer than the other to make it clear which side is which. You can divide the long side of the medium triangle by its unknown short side. Then look at the smallest triangle. The long side of the small triangle is the unknown side, and we can divide that by the known short side of the small triangle. The length of segment DB divided by the unknown length is the same as the unknown length divided by AD. Cross-multiplying and taking the square root provides the answer. If you used two places after the decimal point, your measured value should be accurate to one place after the decimal point.

Thales and the Pyramid

Thales was such a clever man that he became a legend in the ancient world. As a result, some of the stories that were told about him may not actually be true, but we are fairly certain that he amazed the Egyptians by measuring the height of the Great Pyramid. If you do an online search for Thales you will likely find several stories about how clever he was. Thales used a proportion to determine the height of the pyramid. He realized that the ratio between the height of the pyramid and its shadow would be the same as the ratio between the height of his stick and the stick's shadow. He measured the shadow of the stick and divided that by the height of the stick. He either memorized or wrote down his answer. He also measured the shadow of the pyramid. The length of the pyramid's shadow divided by the height of the pyramid should give that same answer. Shadow ÷ height = answer. We know that 6 ÷ 3 = 2, and 6 ÷ 2 = 3. Therefore if Shadow ÷ height = answer, then Shadow ÷ answer = height.

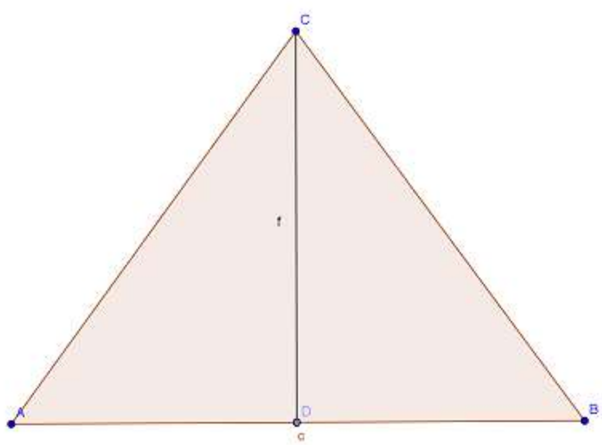
Materials

Protractor
Ruler
Construction Paper
Sunshine

Procedure

We are going to re-create the famous tale of how Thales measured the pyramid, *in reverse*. You will use a small model of the Great Pyramid to measure your own height, in the same way that Thales measured the height of the pyramid. The great Pyramid is really, really big. If you were to use the same ratio between your height and the small model as there is between you and the actual pyramid, your little pyramid would be less than an inch high! It is difficult to get an accurate measurement of the shadow of something so small. Instead, make your model about 10 inches tall. It is a good idea to look up the actual measurements of the pyramid, so your model will have the same width to height ratio as the real pyramid, because you will learn more about proportions that way. Use the height of the pyramid as it was in ancient times. All you need to make a pyramid are 4 triangles and some tape, and a protractor so you can make sure the base angles of the triangles are the same so they will be isosceles. But when you start constructing your model, you'll quickly run into a problem. You are making the slanted sides of the pyramid, but how will you get it to be the height you want?   Fortunately this problem was already solved long before Euclid's time, by Pythagoras sometime around 550 BC – see "A Trick with Triangles: The Pythagorean Theorem."

Once you have decided on the height and width of your pyramid, use the Pythagorean Theorem to calculate how long the hypotenuse (the slanted side) should be. A cross-section of a pyramid is shown below, so you can see how they Pythagorean Theorem applies to this situation. The slanted side of the cross-section will be the height of your 4 triangles.



When you know your measurements, [1]record them and make your pyramid out of construction paper.

Cut out your triangles and tape your pyramid. Now wait for a sunny day, and work either early in the morning or later in the afternoon when shadows are long. That is important here because the little pyramid is small enough that even a slight error in measurement can be a large percentage of its shadow. The shadow should be as long as possible to make the error smaller relative to the total length. Remember that some of the pyramid's shadow is in a sense hidden underneath it. If you were using a little stick with the same height as the pyramid, you would measure the shadow from the base of the stick. Add 1/2 the length of the side of the pyramid to the length of its shadow so that you are really measuring the length of the shadow from just underneath the top of the pyramid.

For your own shadow, stand where the end of it touches some easily identifiable mark, and then mark the spot where you are standing, at about the middle of your feet. [2]Record your measurements.

Analysis

[3]Use the measurements of the shadow of the little pyramid to calculate how tall you are. [4]How close did you get?

What Thales was able to see was that the top of the pyramid, the base of the pyramid, and the tip of the shadow formed a triangle created by the slanted rays of the sun. Using his stick, he created a much smaller triangle with the same angles, using the same slanted rays of the sun. As we saw before, when two triangles have the same angles but are different sizes, their sides are proportional to each other. Therefore the shadow of the pyramid was in the same proportion to its height as the shadow of the stick was to the height of the stick.

Compass Construction: A Perpendicular Line

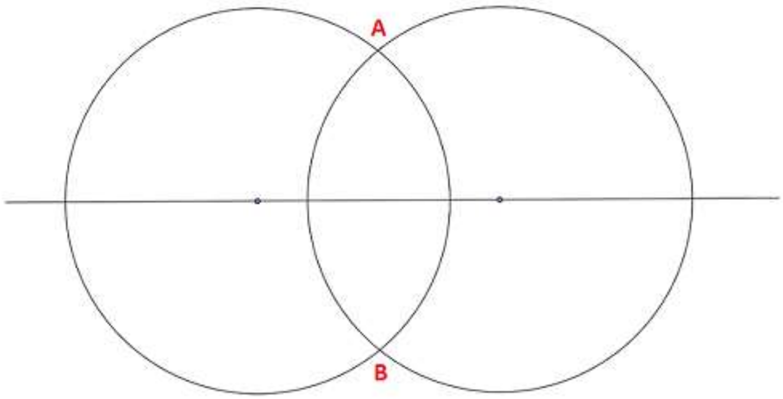
The radius of a circle is a line from the center of the circle to the edge. When you draw a circle with a compass, the center is the small hole in the paper made by the sharp point of your compass. You can draw a radius from this point to anywhere on the edge, and its length will always be the same. Any radius of a circle is the same size. In fact, this is how a compass works, by measuring out a radius of the same size everywhere from the center point. Although it is not particularly amazing that all radii of a circle have the same length, we can use this fact to construct a perpendicular line with a compass. Two lines that are perpendicular form a "+" sign. They are at right (90 degree) angles to each other.

Materials

Ruler
Compass
Protractor

Procedure

Draw a straight line that is about 6 or 7 inches long. Place the sharp point of your compass on the line, and draw a circle. You now have one circle that has its center on the line. Next, we are going to draw a second circle with its center on the line that intersects the first circle. Without changing the size of your compass, move the sharp point a short distance further down the line and draw another circle. You should have two circles of the same size that intersect.



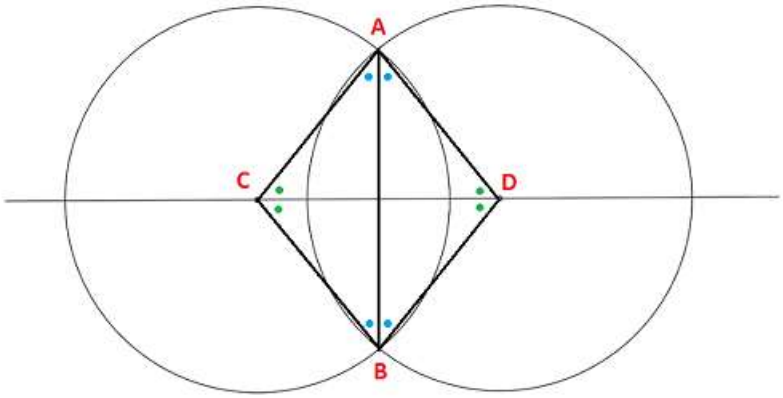
Notice that the circles intersect, or cross each other at two points, A and B. Take your ruler and draw a line through these points.

Now you have two lines on your paper. Take out your protractor and measure the angle between these two lines. If you have worked carefully and precisely the angle should be a perfect 90 degrees. The two lines are perpendicular.

After you have done this a few times, you might not draw the whole circles anymore because you only need the intersect points. You may also want to learn how you can control where the perpendicular line appears. Watch a video here:
http://www.youtube.com/watch?v=d62zCNZP0wg, and a slideshow here:
http://www.mathopenref.com/constperpextpoint.html.

Analysis

The symmetry in the drawing is responsible for creating the perpendicular line. Because all of the radii of the two circles are the same length, there are equal distances between the center of each circle and points A and B.



Line segments AC, AD, BC, and BD are all the same length. Triangles ABC and ABD are identical except for the fact that they are mirror images of each other. Because they share a common side, and their two other sides are the same size, we say that they are congruent by SSS (3 equal sides; see the experiment "Sides of a Triangle"). When triangles are congruent, their corresponding angles are the same. Triangles ABC and ABD are also isosceles triangles, which means that their base angles are the same (see "What is an Isosceles Triangle?"). I have summarized these facts by placing little blue dots in the angles that are the same size.

Next, consider triangles CBD and CAD. They are also congruent and isosceles, which allows us to say that the angles with the green dots are the same.

Line segment CD cuts triangles CBD and CAD into 4 smaller triangles. Each of these triangles has an angle marked with a green dot, and an angle marked with a blue dot. Because we know that the angles of a triangle always add up to 180 degrees, the unmarked angle has to be the same size in all four triangles. Together, these unmarked angles add up to a full 360 degrees, which means that each one of these angles has to be 90 degrees. The line segments CD and AB are perpendicular.

What Makes a Parallelogram?

A parallelogram is defined as a quadrilateral (a shape with four sides) that has two pairs of parallel opposite sides. We usually think of a slanted kind of shape when we think of a parallelogram, but squares and rectangles are also parallelograms because their opposite sites are parallel.

Materials

Geogebra

Procedure

1. Is a Rhombus a Parallelogram?

A rhombus is any shape with 4 equal sides. A square is a special kind of rhombus that has right angles. Is any rhombus a parallelogram?

Draw a random rhombus in Geogebra, using the polygon button. Make sure it is not actually a square. Mark the lengths of the sides (distance in the dropdown menu of the angle button), and move the vertices until the sides are of equal length. Note: for this experiment to work, the distances must be very close to equal so try to get them within .01cm of each other. To check if this is a parallelogram, we need to know if both pairs of opposite sides are parallel. Fortunately Geogebra has a button that lets you draw a parallel line. It is found in the dropdown menu of the perpendicular line button (button 4). Simply draw a line parallel to one of the sides to a point on the opposite side (one of the vertices). If your parallel line matches the other side perfectly, the two sides are parallel. Check both pairs of opposite sides before you draw your conclusion. Is it a parallelogram?

2. If one pair of opposite sides of a quadrilateral consists of parallel line segments that are both 7 cm long, is it a parallelogram?

Draw a four-sided polygon in Geogebra. Draw a line parallel to one of the sides through a vertex of the polygon. Adjust the sides so that you have two parallel segments that are 7 units long. Is your figure a parallelogram?

3. If a quadrilateral has a pair of opposite sides that consists of line segments equal in length to each other, and another pair of opposite sides also consisting of segments of equal length, is it a parallelogram?

Here we have two pairs of equal sides, but they are not necessarily parallel. Draw the quadrilateral as described. Is it a parallelogram?

4. If the opposite angles of a quadrilateral are equal, is it a parallelogram?

I found the angles were hard to get equal. This is a long section, so I cheated by drawing the parallel lines first and then adjusting the vertices to get equal angles. You can do the same. Did you create a parallelogram?

5. Do the diagonals of a parallelogram bisect each other (divide each other in half)?

Draw any parallelogram, and connect opposite corners to create the two diagonals. Mark the intersect point by placing a point there, and measure the line segments to see if they are equal.

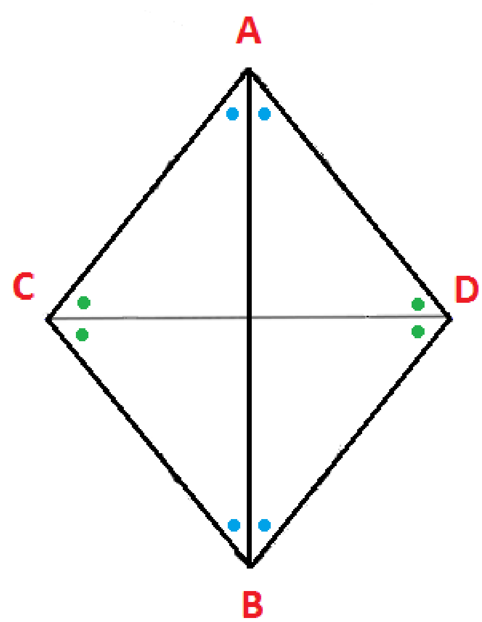
6. If the diagonals of a quadrilateral bisect each other, is it a parallelogram?

To draw this in Geogebra, create line segments for the diagonals. Next mark the intersection of the two diagonals. An intersect point is available from the dropdown menu of button 2. Then measure the distances and get them equal by moving the vertices. Is it a parallelogram?

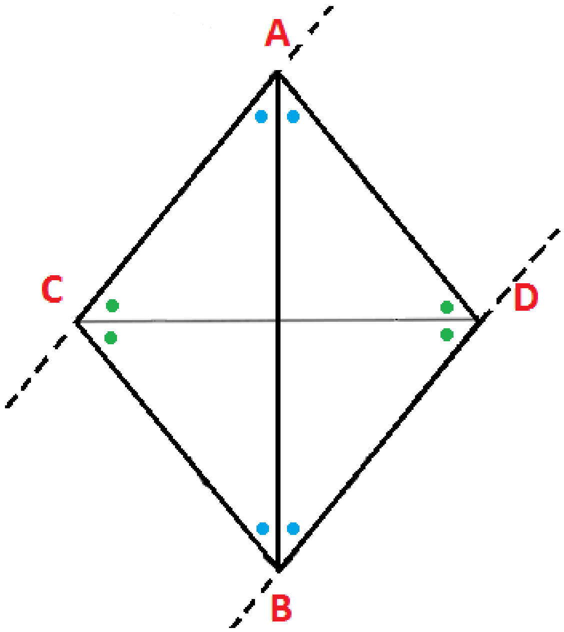
Analysis

1. A rhombus has 4 equal sides. We can draw the diagonals to get sets of isosceles triangles.

If the picture below looks familiar, it should. It is simply recycled from the experiment Compass Construction: Perpendicular Line. I just erased the circles. The explanation can also be recycled.

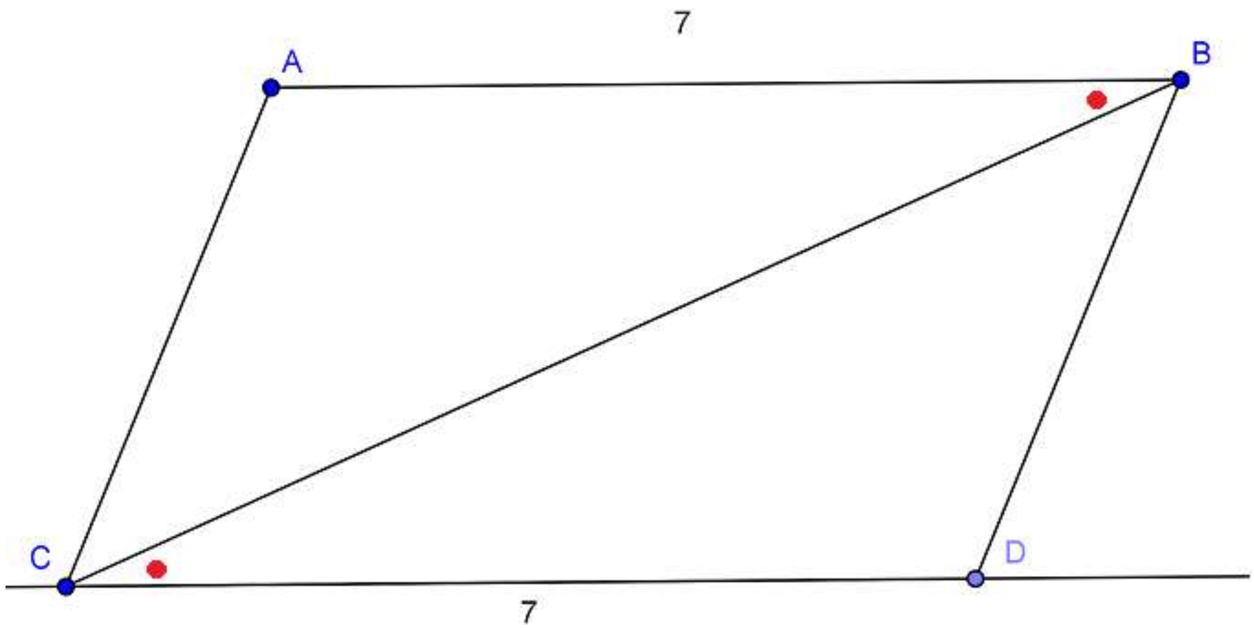


Line segments AC, AD, BC, and BD are all the same length. Triangles ABC and ABD are identical except for the fact that they are mirror images of each other. Because they share a common side, and their two other sides are the same size, we say that they are congruent by SSS (3 equal sides; see the experiment "Sides of a Triangle"). When triangles are congruent, their corresponding angles are the same. Triangles ABC and ABD are also isosceles triangles, which means that their base angles are the same (see "What is an Isosceles Triangle?"). I have summarized these facts by placing little blue dots in the angles that are the same size. Next, consider triangles CBD and CAD. They are also congruent and isosceles, which allows us to say that the angles with the green dots are the same. Then we stop recycling, because we need to draw a new conclusion.



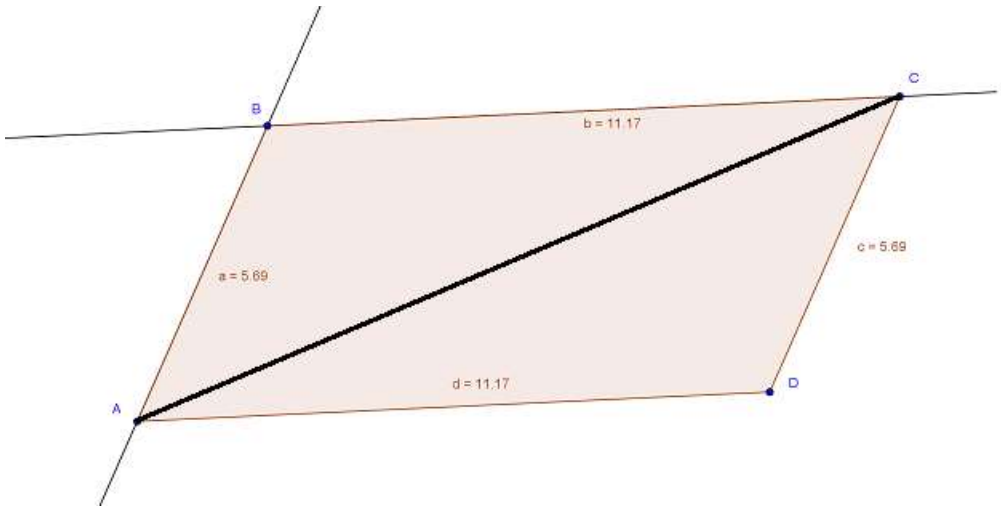
Because alternate interior angles are the same size (blue dots), segments CA and BD are parallel (see Alternate Interior Angles). Also, segments CB and AD are parallel because the alternate interior angles marked with the green dots are equal. The rhombus is a parallelogram.

2. Draw the remaining sides, and one of the diagonals.



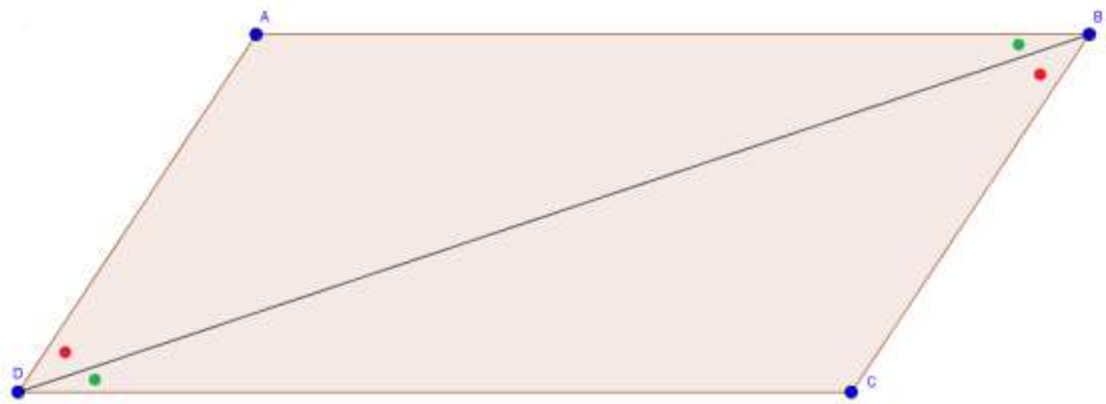
It is given that AB and CD are parallel and of equal length. Because the diagonal BC intersects two parallel lines, it creates alternate interior angles. The two triangles, ABC and DCB, are congruent by SAS. Because we have two congruent triangles, we can say that angle ACB and angle CBD are congruent. These angles are alternate interior angles formed when line CB intersects CA and DB. Because the alternate interior angles are equal the lines are parallel – we have a parallelogram.

3. Draw one of the diagonals. Now you have two triangles that are congruent by SSS (see Sides of a triangle). Because these triangles are congruent, angle BAC is equal to angle ACD. and angle CAD is equal to angle BCA. Because these alternate interior angles are equal, side BC is parallel to side AD and side AB is parallel to side DC. This is a parallelogram.



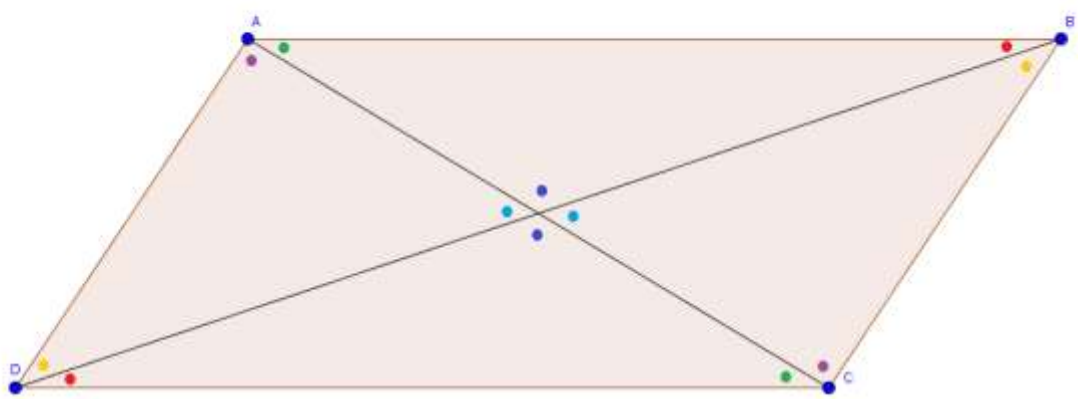
4. Opposite angles are equal in this example. When we have a quadrilateral with angles w, x, y and z, we can say that w + x + y + z = 360 degrees. Here we only have two different angles. Call the largest angle x, and the smallest angle y. x + x + y + y = 360 degrees, or 2x + 2y = 360°. This means that x + y = 180°. For this shape, any two angles that are next to each other (angle x and angle y) add up to 180 degrees. By extending the sides of your quadrilateral a bit you can see that x and y are same-side interior angles. The opposite sides must be parallel, so we have a parallelogram.

5. To prove that the diagonals of a parallelogram bisect each other, we can first show that the two pairs of opposite sides of a parallelogram are always the same length. Draw one of the diagonals, and prove that you now have two congruent triangles (ASA).



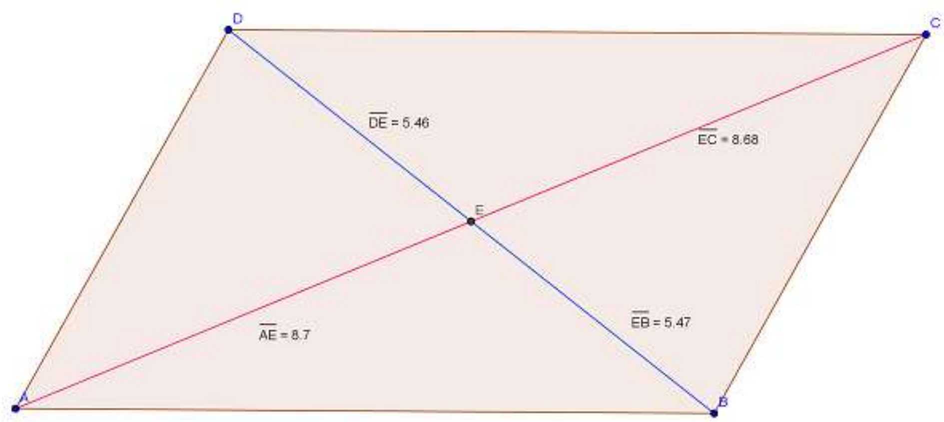
This means that opposite sides must have the same length. Drawing both diagonals creates two pairs of congruent triangles. Because you have already shown that the bases of each pair of triangles are equal, all you need to do is show that they have equal angles.

When you draw the diagonals you create a lot of congruent alternate interior angles, and some vertical angles:



The smaller triangles consist of two congruent pairs. The two sections of each diagonal are equal in length because they are corresponding parts of congruent triangles.

6. You can get nice colors in Geogebra by right-clicking on an object and adjusting its properties. Here, I have colored the diagonals to indicate that the two blue segments are equal to each other, and the two red segments are equal. Brown was the default color, so the brown segments are **not** necessarily equal. If you look at triangles AED and BEC, you can see that they are congruent. They have two the same sides, and the angle between those two sides is the same in each triangle because vertical angles are equal (see Vertical Angles). They are congruent by SAS. Because these triangles are congruent, angle ADE is equal to angle EBC. These two angles form alternate interior angles that show that side AD is parallel to side BC. You can repeat this proof for angles CDE and EBA to show that side AB is parallel to side DC. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.



Assignment

Draw a random four-sided polygon in Geogebra. Mark the midpoints of each side using the Midpoints button from the dropdown menu. Connect these midpoints to create another quadrilateral inside the first. Does it look like that second quadrilateral could be a parallelogram? Move things around – does it still look like the inside figure is a parallelogram?

Divide the outer quadrilateral into two triangles by drawing a line segment between two of the vertices. Explain what creates the inner parallelogram.
(If you need a hint check the experiment "How to Shrink a Triangle".)

The Mysterious Number Pi

The number pi is written as π and pronounced as "pie". This strange number appears when we start measuring circles. When people first thought of the idea of measuring a circle, they may have started by measuring the “width” of the circle. The widest width that you can measure across the circle is called the diameter. The diameter divides the circle exactly in half. The other measurement that you can make is the distance around the circle, which can be found by carefully arranging a string or rope along the outside edge. The distance along the outside of the circle is called the circumference. It probably did not take long for someone to realize that there would have to be a fixed relationship between the diameter and the circumference, so people tried to compare the two quantities. Just how many times longer is the circumference than the diameter? This is how the number π was discovered. But what is so mysterious about π? Well, the circumference is the curved line that you draw to make a circle, and it has a certain fixed length. So does the diameter of your circle. However, somehow when you try to find how many times the diameter fits along the circumference you get a number that can never be expressed precisely.

In this experiment, we will attempt to determine the value of π.

Materials

Large dinner plate with a smooth outer edge
String (about 1/16 to 1/8 inch thick, light color)
Can, preferably a little bigger than a standard small soup can
Permanent Marker
Ruler with cm markings
Geogebra

Procedure

Tie a knot at the end of your string so it doesn't unravel. We will use our string to measure the diameter of the plate. The diameter is the distance across the center of a circle. You'll know when you've found it exactly because it will be the longest of the measurements you make across the plate (Euclid Book III, Proposition 15). Be careful not to pull hard on the string because you will stretch it and get an inaccurate measurement. Use your marker to mark the string at the longest distance across the plate.

Next, we will measure the circumference, which is the distance around the plate. Use small pieces of tape to tape your string all the way around the outside edge of the plate. When you reach the end, there is no need to cut your string. Just mark it with your marker, and cut it later if you want. Carefully peel the pieces of tape away from the string.

Your string now has two marks on it, one for the diameter and one for the circumference. Fold the string accordion style (back and forth) to see how many times the diameter fits into the circumference. You should find that it is a bit more than three times. For a brief moment you will be holding the number π in your hands but you cannot pin down its exact value, just like you can't catch the wind in a jar.

Use the centimeter side of your ruler to measure both the circumference and the diameter. Record your measurements. Divide the length of the circumference by the length of the diameter and record the result.

Next, take a can and measure its diameter with a ruler. Again, the diameter will be the largest possible measurement across the top of the can, so try several times until you are sure you have the longest distance across. Make a mark on the top edge of the can. Put the can down sideways on a piece of paper, and line up the mark with the edge of the paper. Carefully roll the can along until the mark again touches the paper (tape two sheets of paper together if you have a large can). Make a mark on the paper at this point. Measure how far this mark is from the edge of the paper. This is the circumference of the can, neatly rolled out on a piece of paper. Record your measurements. Divide the circumference of the can by its diameter and record the result.

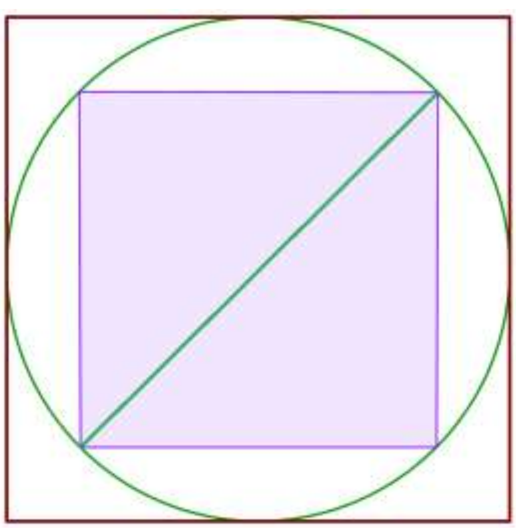
Fiddling around with a string and a can is not extremely precise, and here is where a computer can be handy. Let's use Geogebra to get more accurate measurements. Start Geogebra and maximize the window. Activate the Move Drawing Pad button and click and drag to create a nice clear drawing area. Now activate the circle button. Notice that instructions for creating a circle appear to the right of the buttons. Create a fairly large circle. Activate the angle button and use the tiny arrow in the right lower corner to see the menu. Select distance, then click on the circle to see its circumference. Place a new point, C, on the edge of the circle. Now go to the menu of the line button (the third button from the left) and select segment between two points. Draw a line that starts at the point B on the edge of the circle and ends at point C. Move point C until the line passes through the midpoint, point A. Now segment BC is a diameter of the circle. Activate the angle button menu and measure the length of the diameter. Then divide the length of the circumference by the length of the diameter. Repeat this process twice with different circles. Record your measurements and the results of your calculations.

Analysis

The circumference of a circle is π times the diameter. This is written as $C = \pi D$. Because the diameter of a circle is twice as long as the radius, we can say that $D = 2r$, where r is the radius. In the formula for the circumference, D is often replaced by 2r. Because the convention is to write numbers before letters, the formula looks like this: $C = 2\pi r$. Unfortunately this formula looks very similar to the formula for the area of a circle, and students tend to confuse the two. For this reason it is better to use $C = \pi D$. That also helps you remember where π comes from.

Compare the results of your experiments with the approximate value for pi listed here: http://mathforum.org/dr.math/faq/faq.pi.html

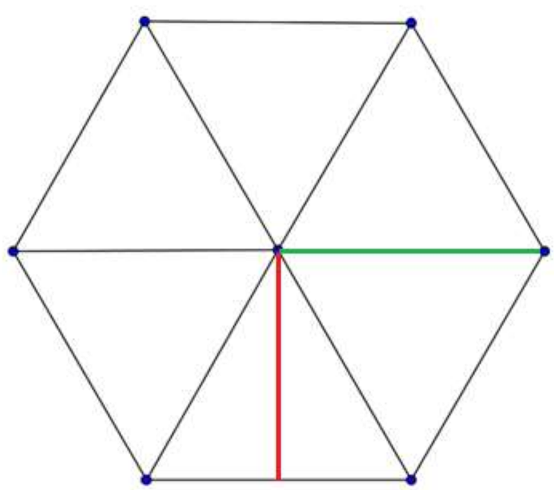
In ancient Greece, Archimedes worked on finding a value for π. His idea was to trap the elusive number π between two known values. The simplest way to do this is to draw a square inside your circle, and another square around the circle, like this:



The circumference of the circle will be somewhere in between the *perimeter* (the distance around the square) of the outside square and the perimeter of the inside square. To find pi, we divide the approximate circumference of the circle by its diameter. The outside square has sides that are equal to the diameter of the circle. Its perimeter is 4 times the diameter, which can be written as 4D. When we divide that by D, we get 4 for the value of pi, which would be too big. The inner square has a perimeter that is smaller than the circumference of the circle. The green line, which is a diameter of the circle, divides the inner square into two right triangles. Because two of the sides of each triangle are the same, and the hypotenuse has length D, we can calculate the perimeter of the inner square by using the Pythagorean Theorem. What is the perimeter of the inner square, in terms of D? What estimate for pi do you get when you divide this perimeter by D? (Show your calculations.) This tells you that the value of pi must be somewhere between 4 and 2.8284.

This rough estimate is obtained by using a square. We can get a much more accurate estimate if we use inner and outer polygons with more sides.

Draw a circle on paper and place a regular hexagon (a polygon with 6 sides) inside of it with the vertices touching the edge of the circle. Then draw a regular hexagon just around the circle. The advantage of using hexagons is that they can be divided up into 6 equilateral triangles.



For the inner (inscribed) hexagon, the green line will be the radius of the circle. For the outer (circumscribed) hexagon, the red line will be the radius of the circle.

Show your calculations to determine a range for the value of pi by finding the perimeter of each hexagon. Use your knowledge of special triangles (see "The Pythagorean Theorem – Special Triangles"), and remember that the radius of a circle is $\frac{D}{2}$. Using hexagons, we find that the value of pi must be between 3 and 3.4641.

Later on, when we look at Trigonometric Ratios, we will use a polygon with 12 sides to get an even better estimate for pi. Archimedes actually used polygons with 96 sides to get a really close approximation for pi. In this way, he determined that the value of pi is just slightly less than $\frac{22}{7}$. Dividing 22 by 7 gives the value of this fraction, which is 3.142857……., with all the digits after the decimal repeating endlessly. The real π however has digits that never repeat because it is **irrational**, meaning that it cannot be written as a ratio like $\frac{22}{7}$.

Practice

Suppose a circle is inscribed in a regular hexagon that has a perimeter of 36 mm. This means that the circle is just touching the midpoints of the sides of the hexagon. Show that the circumference of the circle must be $6\pi\sqrt{3}$ mm, or approximately 32 mm.

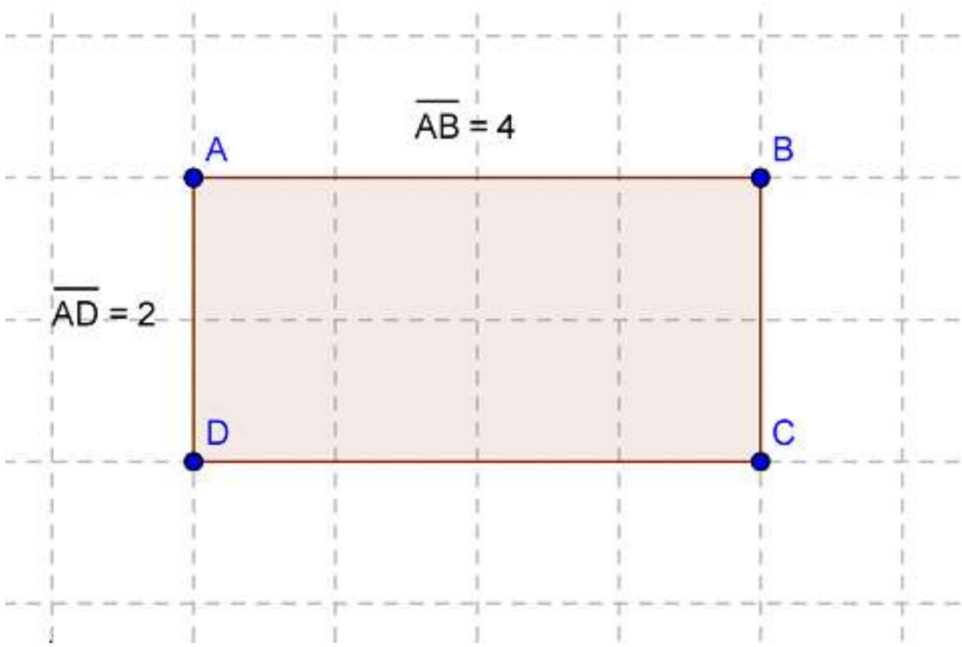
Understanding Area

Hopefully you already have a good understanding of what the area of something is. When I think of area, I think of it as the size of the "flat space" on the ground or on a surface. The tricky part is to measure just how big that area is. Here is a simple problem: find the area of rectangle ABCD below. The length is 4 units, and the width is 2 units. We can just look at the picture to see that the area takes up 8 little squares, or 8 square units. This happens because we can fit 4 little squares along the length, and 2 little squares along the width. That makes 2 rows of 4 squares, which is 8 squares. The area of a rectangle can be found by multiplying the length and the width.

Area = Length x Width

This is often written in the simpler form A = LW. Two letters placed next to each other means that the two amounts represented by those letters should be multiplied.

Caution: some students confuse the area with the perimeter. The perimeter is the distance *around* a figure (2 + 4 + 2 + 4 = 12 units for the figure shown below). The general formula for the perimeter is P = 2W + 2L.



Usually finding the area of a rectangle is not so difficult. However, a problem may give the area of a rectangle, and the width, leaving you to figure out the length. For example: The area of a rectangle is 8.25 square inches, and the width is 1.5 inches. What is the length? In Geometry, and in many other subjects, you will need to be able to rearrange formulas to get the quantity you want to solve for. A = LW can be rearranged so that it says L = …

A quick way to do that is to compare this formula to 6 = 2 x 3. If we want 2, we would divide 6 by 3: $2 = \frac{6}{3}$. Just do the same thing with A = L x W, so you get $L = \frac{A}{W}$. Algebra would tell us to divide both sides of the equation by W, which works out to the same thing but is a little slower. I usually use 6 = 2 x 3, $2 = \frac{6}{3}$, and $3 = \frac{6}{2}$ to rearrange formulas. In this case that looks like A = LW, $L = \frac{A}{W}$ and $W = \frac{A}{L}$.

Once we know that $L = \frac{A}{W}$, we can find the length of the rectangle: $L = \frac{8.25}{1.5}$ = 5.5 inches.

You may be asked to find the surface area of a box. This is not difficult to do, but it can be a bit hard to visualize when the box is just drawn on paper. It is best to grab a real box to practice on, like a cereal box or a tissue box. Notice that your box is made up of 6 rectangles or squares. These six surfaces are called “faces” in geometry. All you need to do is find the area of each face, and then add them all up. This is made easier by the fact that some of the faces are the same. If you look at your box, you will see that the top and the bottom have the same area. So does a side and the opposite side. A cereal box has 3 pairs of faces that you have to find the area of. If your box is a perfect cube, you only have to find the area of one face and multiply that by 6. Find a box and measure the height, width and length. Determine the surface area. Show your calculations.

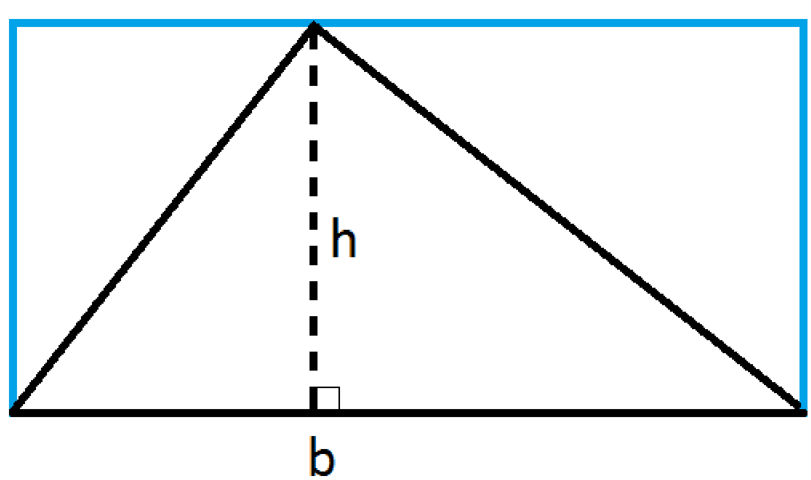
If for some reason you have to find the surface area of a huge number of different boxes, there is a shortcut you can use. To see how it works, find an empty box and remove the top and bottom. Now cut open the remaining part of the box along one of the edges. Spread all 4 side faces out flat, to form a single long rectangle. Notice that the long side of the rectangle is the same as the perimeter of the bottom (or the top), while the short side is the height of the box. You can therefore find the surface area of a box by multiplying the perimeter of the bottom by the height, and then adding two times the area of the bottom.

Area measurements are always stated in square "units", like square feet (ft^2), square inches ($inch^2$ or in^2), square meters(m^2), etc., yet such square units only fit nicely into rectangles or squares. When we have a different shape, we really need to find a rectangle or square that has the same area as that shape. Specific ways of doing that are outlined here for common geometric shapes.

The Area of a Triangle

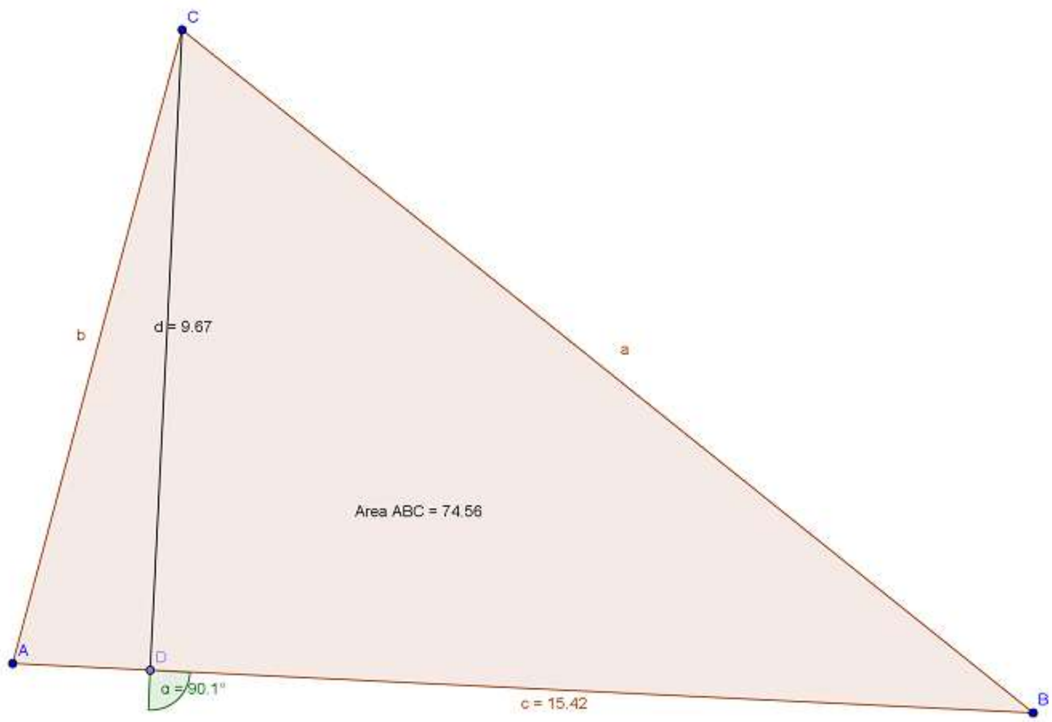
As we have seen, we can take any triangle and construct a rectangle around it (see "The Magic Triangle"). The rectangle will have the same base b as the triangle, and the same height h. The dotted line is the **altitude** (a fancy word for height), and it is always drawn at a 90 degree angle to the base. If you look carefully, you can use congruent triangles to prove that the rectangle must have twice the area of the triangle. This leads to the formula for the area of a triangle:

Area is half the base times the height; often written as A = ½ bh

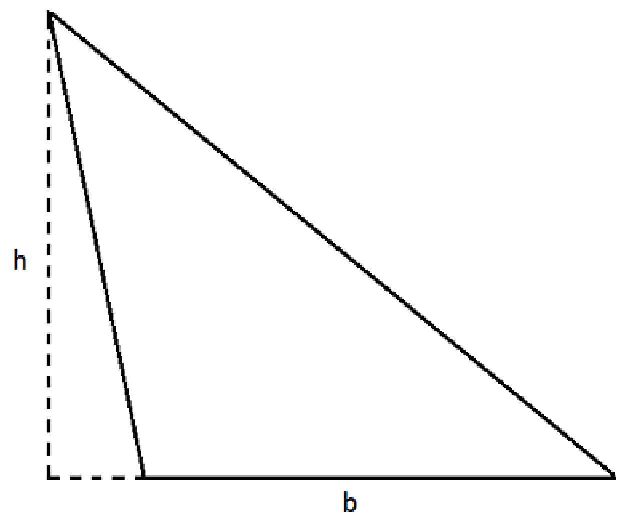


Try it out for yourself in Geogebra, as shown in the figure below. You will need to draw a triangle using the polygon button, and place an extra point D on the base. Create a line segment between D and C, and measure its angle by first clicking on the segment and then on the base of the triangle. Move point D until the angle is as close to 90 degrees as possible.

Then use the dropdown menu on the angle button to get the base and the height of the triangle, and also its area. Do your own calculations - do you get the same area as Geogebra does?

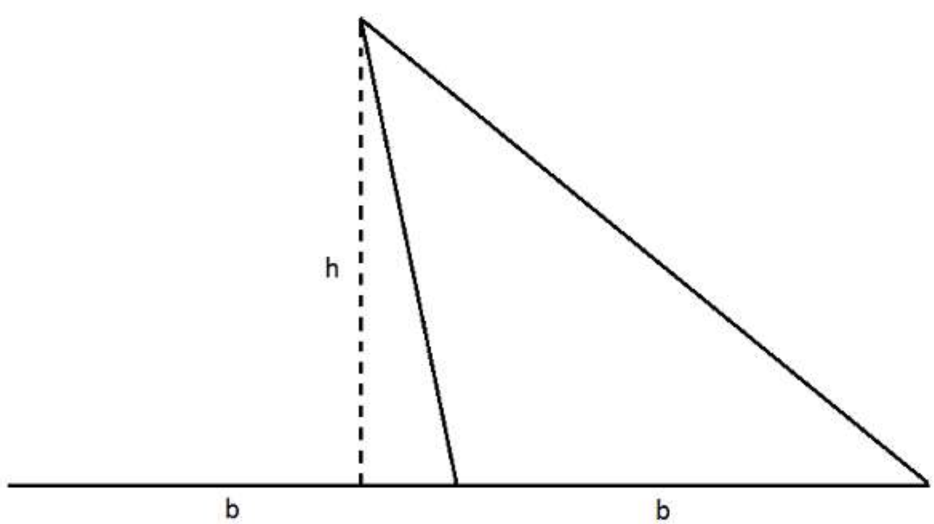


Sometimes you are given the height of a triangle in a different way:

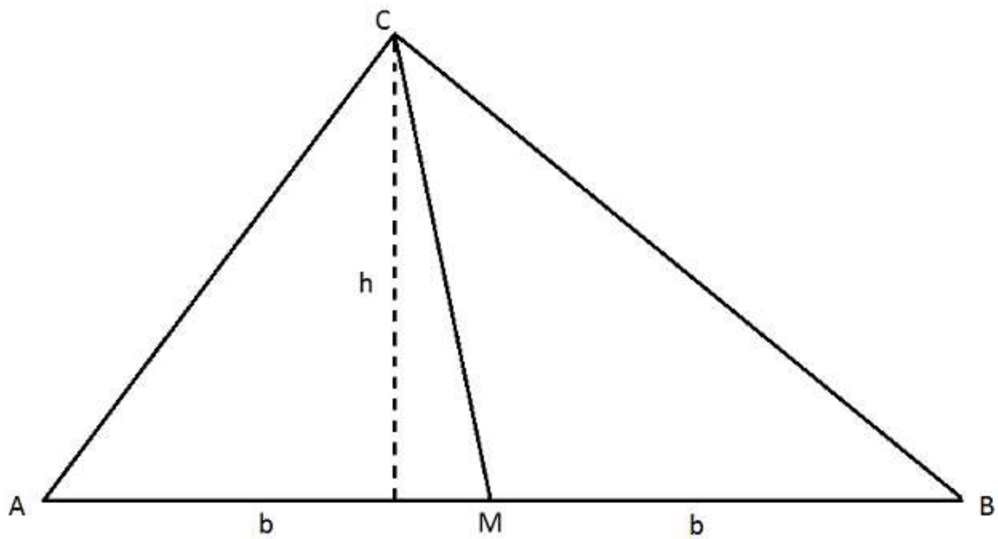


Can we still use the regular formula for the area? Let's see. Draw your own similar triangle on paper using some arbitrary values for b and h, like maybe 5 inches and 4 inches. First, calculate the area of your triangle using the formula ½bh.

Next, we will see if we can find the area another way. Extend the base of the triangle by a length equal to the original length b, as shown below:



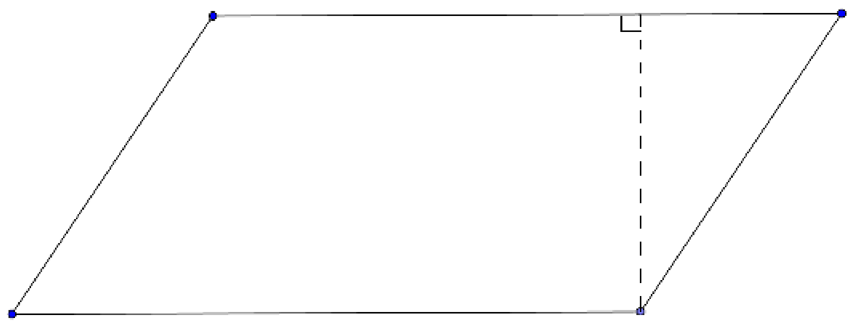
Then create triangle ABC by drawing one more line, like this:



Calculate the area of the large triangle, triangle ABC. It has height h and base 2b. Triangle ABC is made up of two smaller triangles, your original triangle (now labeled MBC) and the new triangle AMC. Calculate the area of triangle AMC. It has base b and height h. Take the area you calculated for the large triangle ABC, and subtract the area of triangle AMC. This should leave you with the area of your original triangle. You should get the same result as when you use the formula ½bh to find that area. This tells us that we can use the formula even when we measure the height of a triangle outside the triangle itself.

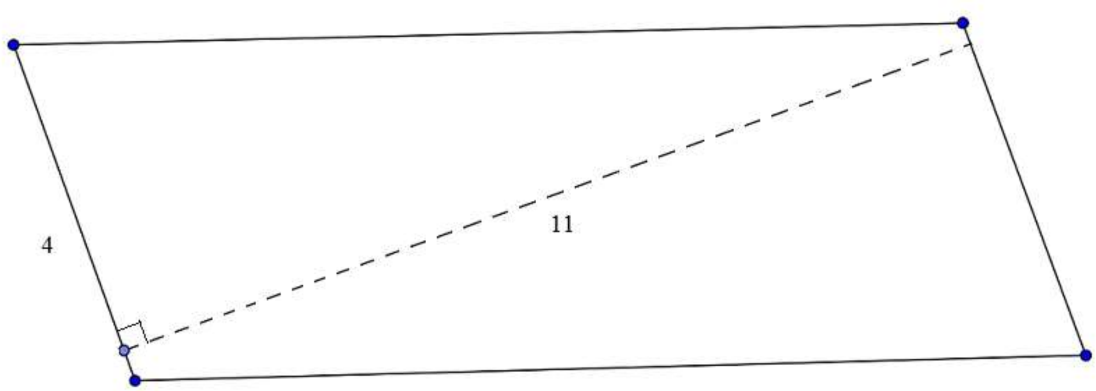
The Area of a Parallelogram

Opposite sides of a parallelogram are parallel to each other. This causes some of the involved angles to be equal to each other, which makes it easy to change this shape into a rectangle so we can measure its area. First, we need to draw a parallelogram. Use construction paper. Carefully construct two parallel line segments **of equal length** using your compass and a ruler. Directions for constructing a parallel line can be found at http://www.mathopenref.com/constparallel.html. Once you have the first two segments placed properly, you can just connect their endpoints to complete the parallelogram. Next, draw a perpendicular line from the top segment to one of the vertices as shown below:



Now cut out your shape, and then cut along the dotted line. Move the cut-off triangle to the left side to make your shape into a rectangle. Does it fit just right? Look back at the experiment "What Makes a Parallelogram" to see why it does. The formula for the area of a parallelogram is simply the base times the height, just as it is for the rectangle that you can make from it. Write your own explanation for why this works.

Sometimes the area of a parallelogram is provided “sideways” like this:



The area is still the base times the height. Here the base is 4 and the height is 11, so the area is 44 square units.

The Area of a Circle

To follow the math in the formula, you need to know the notation for squared numbers. 5^2 means 5 x 5. Any number squared is just that number times itself. r^2 means r x r.

Materials

Geogebra

Compass

Ruler

Scissors

Construction Paper

Glue

Calculator

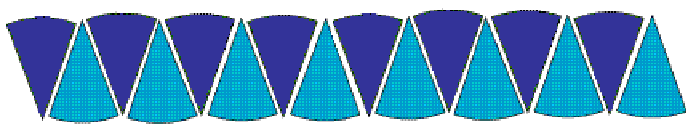
Procedure

Use your compass to draw a large circle on a piece of construction paper. Use a ruler to draw a diameter, which is a line that extends from one edge of the circle to another and includes the center point. (The center point is the hole made by the sharp point of your compass.) Measure the length of the diameter and record it. Next, we are going to use our compass to divide the circle into small pie-shaped sections.

First, adjust the distance between the arms of your compass to about 2 inches. Then place the sharp point of your compass back in the center hole of the circle, and make a mark on the diameter line on either side of the hole. You should now have two points, which we will call P and Q, that are located on the diameter line at equal distances from the center. We will use these points to construct a new diameter that is perpendicular to the diameter you drew. (Click on the link and watch the demonstration).

Now our circle is divided into 4 equal sections. We can divide it further by bisecting angles. Bisect means to cut into two pieces. Watch a demonstration here: http://www.mathopenref.com/constbisectangle.html. At first you have 4 large angles to bisect. Do them all, and use the points you create to draw two new diameters for the circle. Now we have 8 pieces. Divide them again so that your circle has 16 pieces.

Take your scissors and cut along the lines to get 16 separate pie-shaped pieces. Take out a second sheet of construction paper. We are going to glue all our pieces to this construction paper, kind of like this except with no spaces between them:



This gives you a shape that you can use to estimate the area of the circle. This shape is close to a parallelogram. Notice that its length is approximately one-half of the circumference of the circle, and the height is approximately equal to the radius of the circle (the radius is 1/2 the diameter, which you measured earlier). Measure the height and approximate length of the shape you made and record your measurements. Now calculate the area and record it.

Analysis

The actual area of a circle is given by the formula Area = πr^2 (π times r times r), where r is the radius. Using a value of 3.14 for π, calculate the actual area of your circle. How does that compare to your measured area?

Now imagine that you would make your pieces even smaller, until they are really tiny slivers. The smaller you make your pieces, the closer to a rectangle you get. The height of the rectangle would be nearly exactly the radius of the circle. The length would be very close to half the circumference (half the pieces make up the length). When you calculate the area of this rectangle, you multiply the height, which is r, by 1/2 the circumference, which is r times π. So, when your pieces are infinitely small, the area is exactly πr^2!

We have just seen how we can cut up a circle into infinitely many infinitely small slivers to make a rectangle with an area of πr^2. However, we also have a very similar-looking formula for the circumference of a circle, which is $2\pi r$ (better written as $C = \pi D$). How will you know which one to use when the time comes to find the area of a circle?

Actually, there is an easy trick for that. Normally we leave units like inches or centimeters out of our calculations because they just clutter things up. When you go to use the circle formulas try leaving the units that you are given. For example: A circle has a radius of 5 centimeters. What is the area of this circle? Draw the circle in Geogebra. You will need to create a line segment between the center of the circle and the point on the edge. Use the menu on the angle button to measure the length of the segment. Then adjust your circle until the radius is 5 cm.

$2\pi r$ would give us 2π times 5 centimeters, which is 10π centimeters. Because the unit in the answer is "centimeters", it gives us a strong hint that we have calculated a length rather than an area. This was the formula for the circumference of the circle.

πr^2 gives π times 5 centimeters times 5 centimeters which is 25π centimeters^2 or 25π square centimeters. That is an area, so πr^2 must be the right formula to use in this case. Use a calculator to get the answer ($\pi = 3.14$). Then find the area in Geogebra. (Area is in the angle button menu). Does the measured value agree with your calculations?

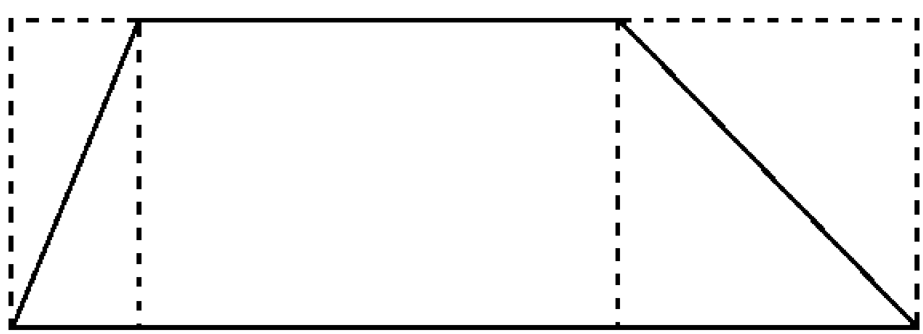
Practice

When you draw the largest possible square inside a circle, the square is said to be inscribed in the circle. The vertices of the square touch the edge of the circle. Suppose a square inscribed in a circle has an area of 36 cm^2. Show that the area of the circle is 18π. (Look back at "Special Triangles" in the experiment "A Trick with Triangles if you need help.)

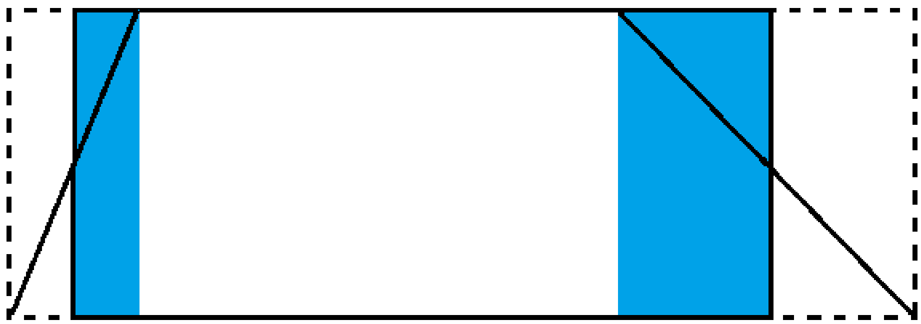
The area of a Trapezoid

A trapezoid is a shape with two of the sides parallel to each other. The parallel sides are called the **bases**. The other two sides can stick out (or in) at whatever angle they want to. Below is a picture of a random trapezoid. As the dotted lines indicate, it can be divided into a rectangle and two triangles.

We know that the area of a triangle is half the area of the rectangle that we can draw around it.



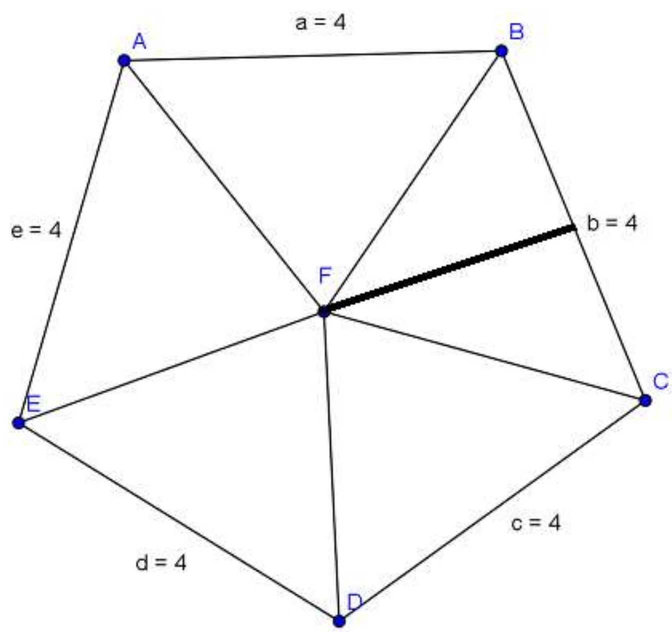
Therefore, we can just replace the triangle with half of the rectangle as shown below:



Now we have a rectangle with the same area as the original trapezoid. Notice that in order to make this rectangle, we have removed some of the length of the bottom side of the trapezoid and added it to the top, so that both bases are equal. That is the same as taking each one of these sides and making it equal to the average of the two sides. The formula for the area of a trapezoid is: the height times the average length of the two bases. Show that the area of a trapezoid with a height of 5 inches and bases of 8 inches and 11 inches is 47.5 cm^2. If you are not sure how to find the average of two numbers, check the e-book "Arithmetic Review".

The Area of a Regular Polygon

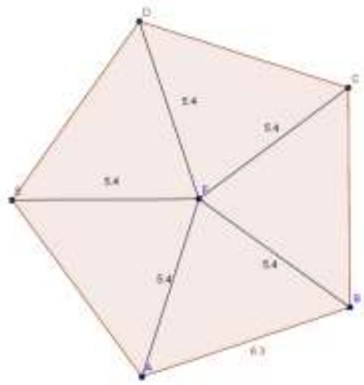
Regular polygons can be conveniently divided into triangles so that we can calculate the area. All you need is the length of the sides of the polygon, which form the bases of the triangles. You also need the height of one of the triangles, which would be the bold line in the regular pentagon below:

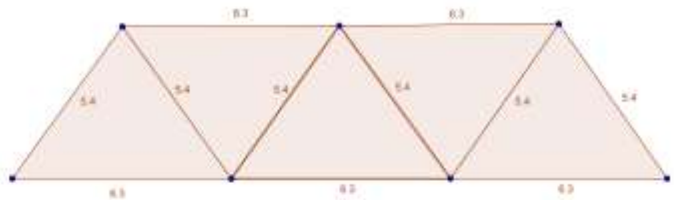


It would not make sense to call the bold line the "height", since it is not the height of the pentagon. The height of an individual triangle in a polygon has been given a special name. It is called the **apothem**. If you know the length of a side, and the length of the apothem, you can find the area of each triangle by taking 1/2 the apothem times the side. Then you have to multiply that by the number of triangles, which is exactly the same as the number of sides.

If the regular pentagon above has an apothem that is 2.75 cm long, show that its area must be 27.5 cm^2.

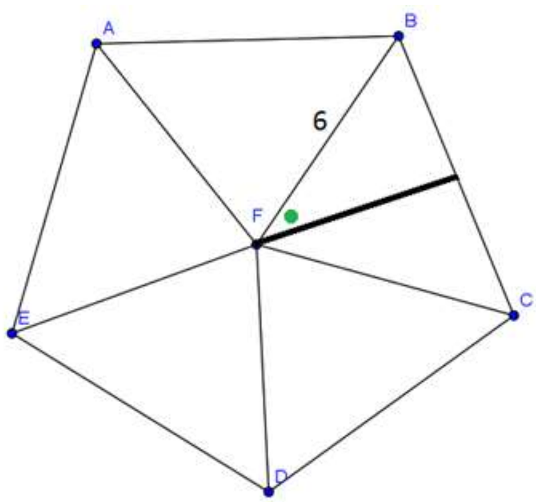
If you have a lot of problems that ask for the area of a regular polygons you may want to use a shortcut. Simply multiply 1/2 times the apothem times the perimeter (the distance all the way around, which is the length of one side times the number of sides). A good way to see that this works is to cut up your pentagon into its component triangles and arrange those triangles to create a trapezoid. The area of the trapezoid will be the top length plus the bottom length (the perimeter of the pentagon) divided by 2, times the apothem:





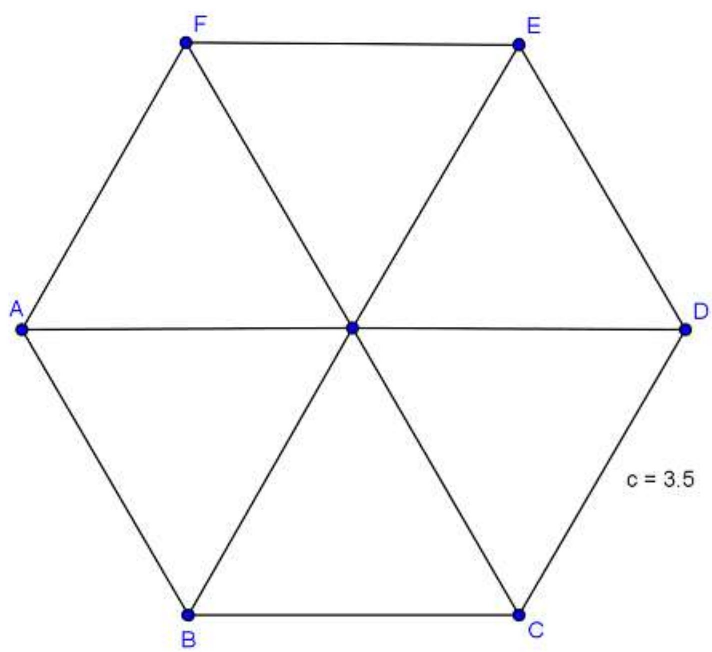
Any regular polygon can be turned into either a trapezoid or a parallelogram this way. (Credit for this idea goes to Kellen C., grade 9). Use this method to find the area of the regular pentagon shown here, which has a perimeter of 6.3 cm times 5 = 31.5 cm, if the length of the apothem is 4.3 cm. Your answer should be about 68cm^2.

Advanced: If you have been learning about trigonometric ratios (like tangent, sine and cosine), you may be slightly dismayed when the nice problem above is given to you like this:



Notice that neither the length of the apothem nor that of the base is supplied. However, if you can figure out the angle with the green dot, you can use your trigonometric ratios to find the missing length. The size of this angle is not that difficult to determine. It is just 1/2 the size of the angle closest to the center of each one of the five triangles that make up the pentagon.

How big are these angles? Well, there are 5 of them, and together they make up 360 degrees. Therefore each of these angles has to be 72 degrees, which makes the angle with the green dot 36 degrees. The sine and cosine of 36 degrees will help you get the required values. The apothem is about 4.854, and the perimeter of the pentagon is 5 times 3.5267 times 2, or about 35.27. That makes the area of this pentagon about 85.6 square units.



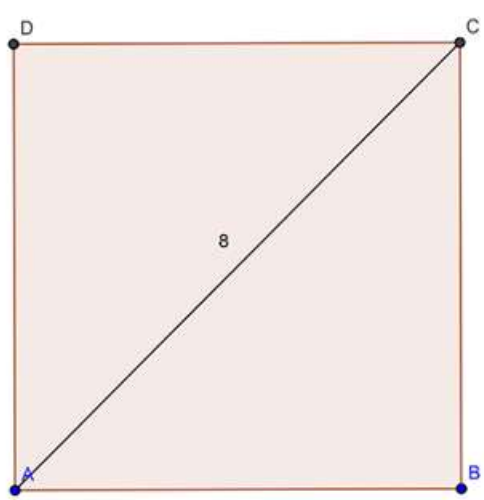
This is a hexagon (a regular polygon with six sides). What is its area? You may be puzzled by this question, since only the length of a side is given. To see what is happening here, it is best to create your own hexagon.

Use a protractor and a ruler, and draw on construction paper. Create a triangle that has three 60 degree angles. Measure the sides of the triangle. Are the sides equal? Why do you think this is an equilateral triangle?

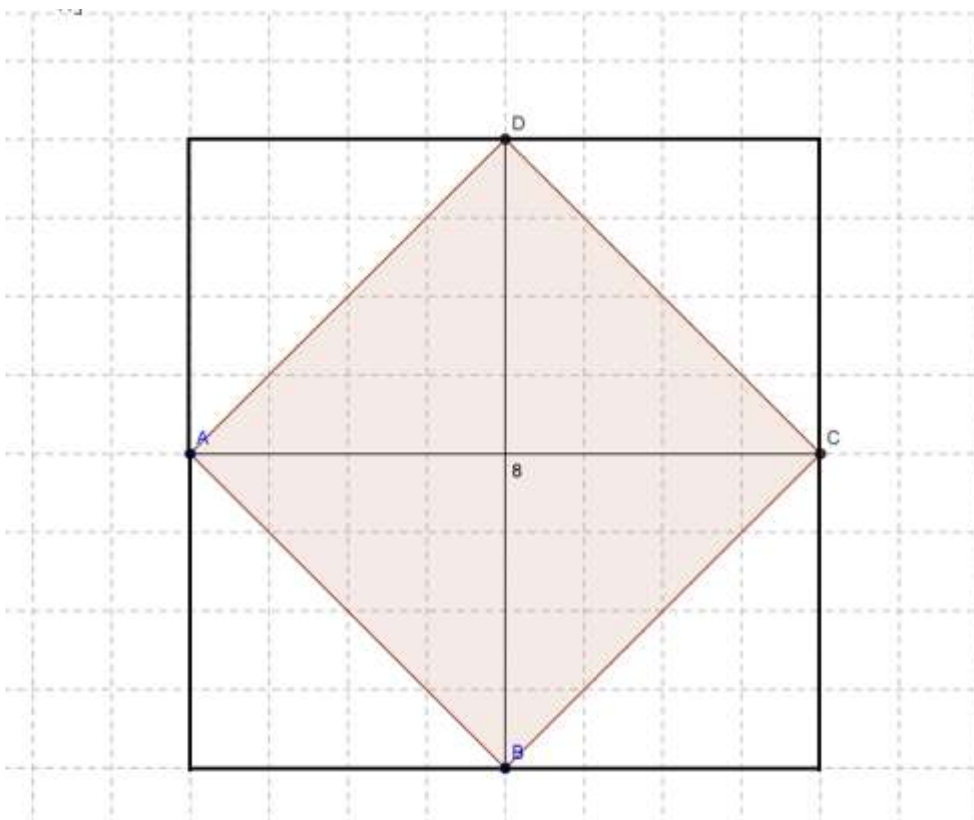
Cut out your triangle and use it as a template to trace 6 identical equilateral triangles. Arrange your triangles to make a hexagon. What is the sum of the angles of a hexagon? What is the measure of each individual angle? Show your calculations. Why do 6 equilateral triangles make a regular hexagon?

Take one of your triangles and make a mark at the exact middle of one side. Draw a line from this point to the opposite vertex. This is the apothem, or the height of the triangle that we need to calculate the area. Cut your triangle in half along this line. You should now have two identical triangles that each have one 90 degree angle. These are right triangles, and their base is half the length of the side of the original triangle. If, as in the example above, the original triangle had sides of 3.5, our new triangles have sides 1.75, 3.5, and an unknown side. We can use our knowledge of 30-60-90 triangles (See "A Trick with Triangles: The Pythagorean Theorem) to quickly find the unknown side, and so determine the area of the hexagon pictured above.

Finding Areas from Given Diagonals

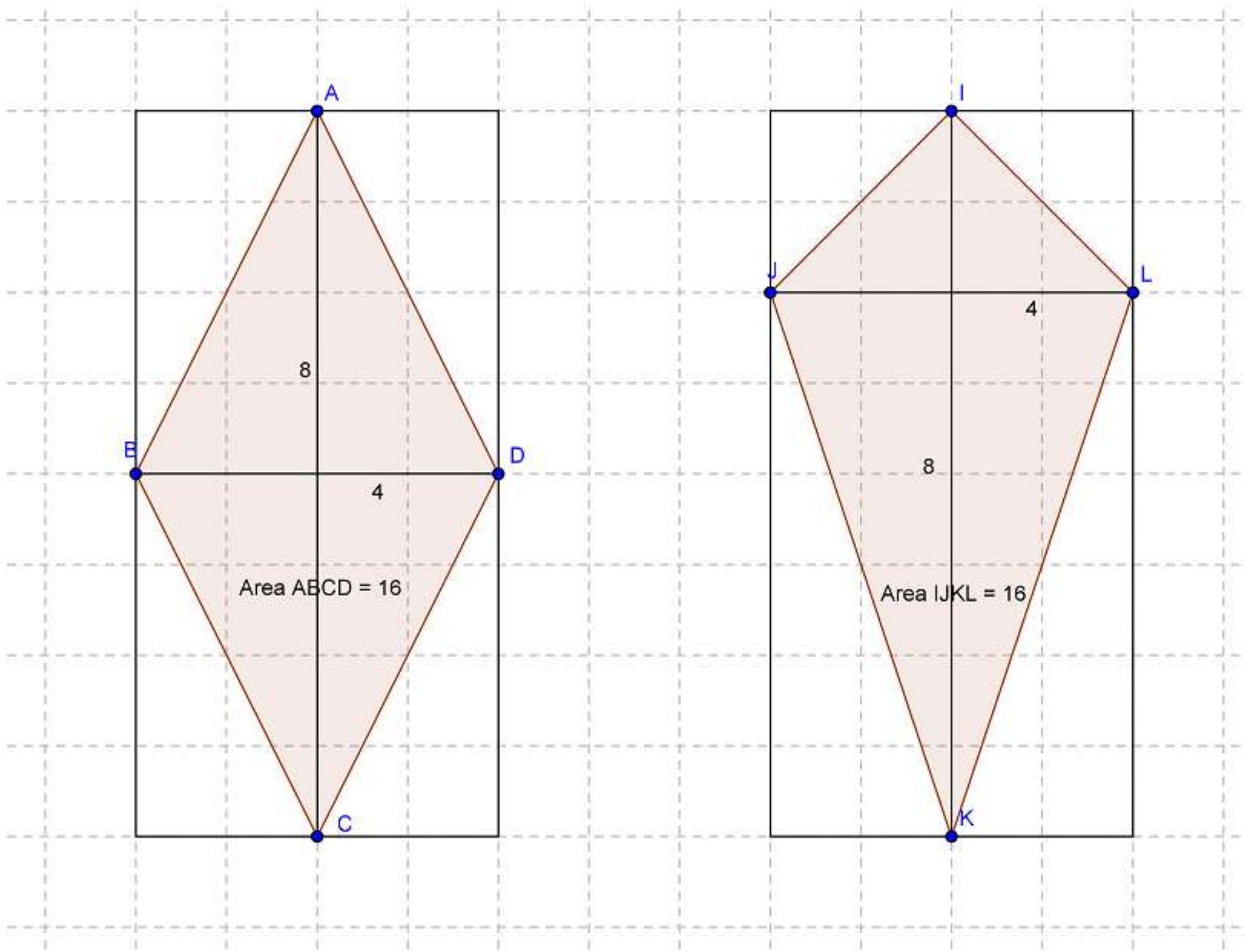


Here is another regular polygon that happens to be a square. If we knew the length of the sides the area would be easy to find, but unfortunately we only have the length of the diagonal, which is 8 cm. This problem is easily solved by turning the figure so that one of the corners points up. Now draw a larger square around it that just touches all four corners:



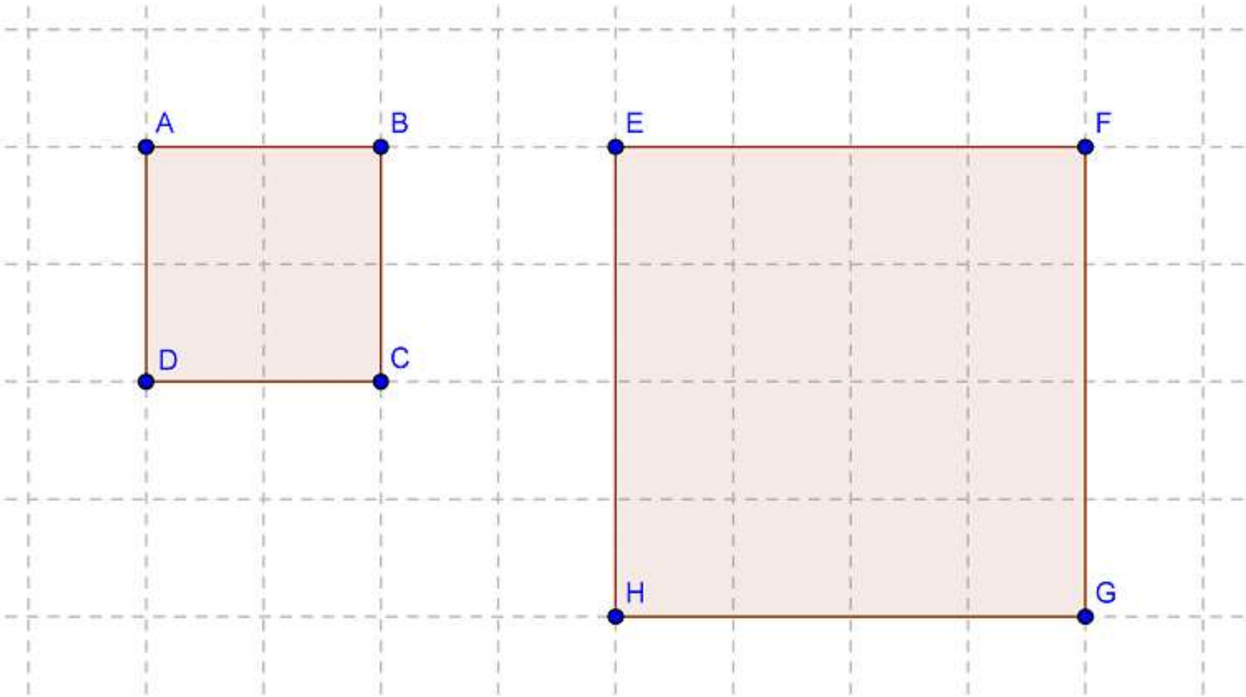
The sides of the larger square are the same length as the diagonals of the smaller square. That means that the larger square has an area of 8 cm x 8 cm or 64 square cm. Because of the symmetry of the figure, we see that the original square is exactly half the area of the larger square, so it must be 32 cm^2. There is an actual formula that says that the area of a square is one half times the product of its diagonals, but a simple picture is easier to remember.

This trick also works for two other symmetrical figures that have their diagonals at 90 degree angles to each other, the rhombus and the kite:



Enlarging Shapes

Suppose you have a square with sides that are 2 inches long. How do you make this square twice as big? The first thing you might try is to double the sides, to 4 inches. Draw the original square, and the new square that has sides of 4 inches. How many square inches can you draw inside these squares? What happened to the area when you doubled the sides?



Geogebra allows you to quickly compare areas of different shapes. Try doubling the sides of a rectangle to see what happens to the area. Next try multiplying the sides by 3 and measure the area again. Also try the same thing with a triangle, and a random polygon. What happens to the area if the sides are made three times as long?

Analysis

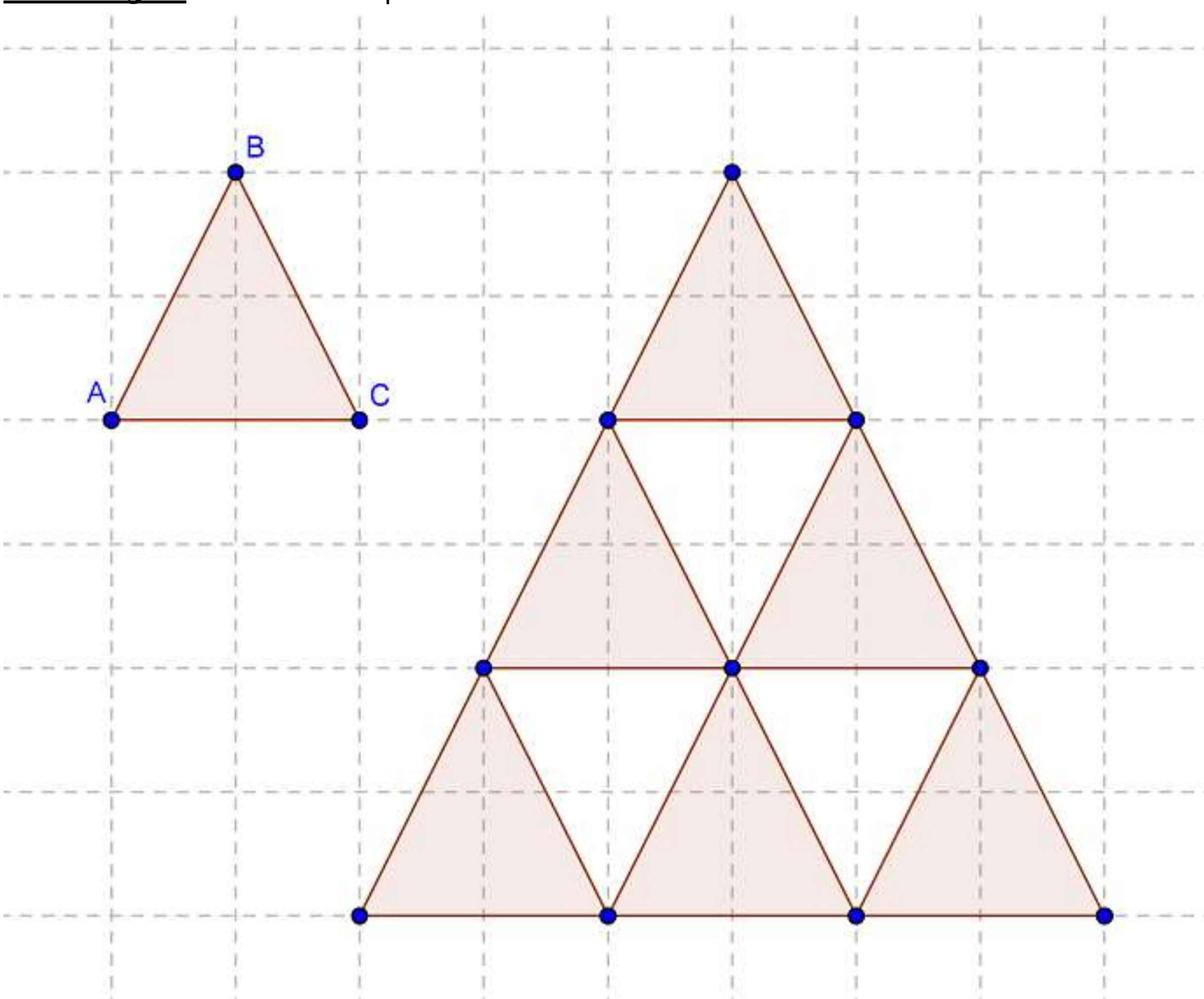
1. Squares and Rectangles

The area is the product of the sides. The area of the original shape is the length times the width = area. When you double the sides, the new area is 2 times the length times 2 times the width = 4 times the area. If we triple the sides we get 3 times the length times 3 times the width = 9 times the area. In algebra notation, it would look like this:

Call the width W and the length L. The area is WL. If you triple the length of the sides you get a width of 3W and a length of 3L. The new area is $3W \cdot 3L = 9WL$, which is 9 times larger than the original area.

If you double the sides of a square, the area gets 4 times as big. How can you make a square that is twice as big? Well, you should multiply the sides by something, like this: "something times the length" times "something times the width" = 2 times the area. The only thing that works for "something" is the square root of 2, since $\sqrt{2}$ times $\sqrt{2}$ is equal to 2. Multiply the length of the sides by $\sqrt{2}$. Try that out in Geogebra to make sure it works for you. Record your data and show your calculations.

2. Triangles Consider the picture below:



This shows what happens if you make the sides of triangle ABC three times as long. Nine triangles fit inside the new triangle, so the area is 9 times as big. We find the area of a triangle by multiplying 1/2 the base times the height. if the base is 3 times as big, and the height is also 3 times as big, then the new area would be 1/2 times 3 times the base times 3 times the height, which is 9 times the area. But how can we be sure that the height of the triangle will be 3 times as big if we triple the sides? In the experiment "Angles of a Triangle", we saw that if two triangles have the same angles they are similar, which means that they have the same proportions and the sides are in the same ratio. In the same way, if the sides of two triangles are in proportion, their angles are the same and they are similar. All of their proportions are the same, so the height will also be 3 times as big if the sides are tripled.

3. Polygons A random polygon can be divided into triangles so that we can calculate its area. If we double or triple the sides, the sides of these triangles will also increase proportionately, with a predictable effect on the total area.

Going in Circles

You already know how to draw a circle using a compass, but when you want to draw one using computer software you may need to understand more about the mathematics of circles. Fortunately the Geogebra programmers have already done the hard work for us, and we can just look at what they did.

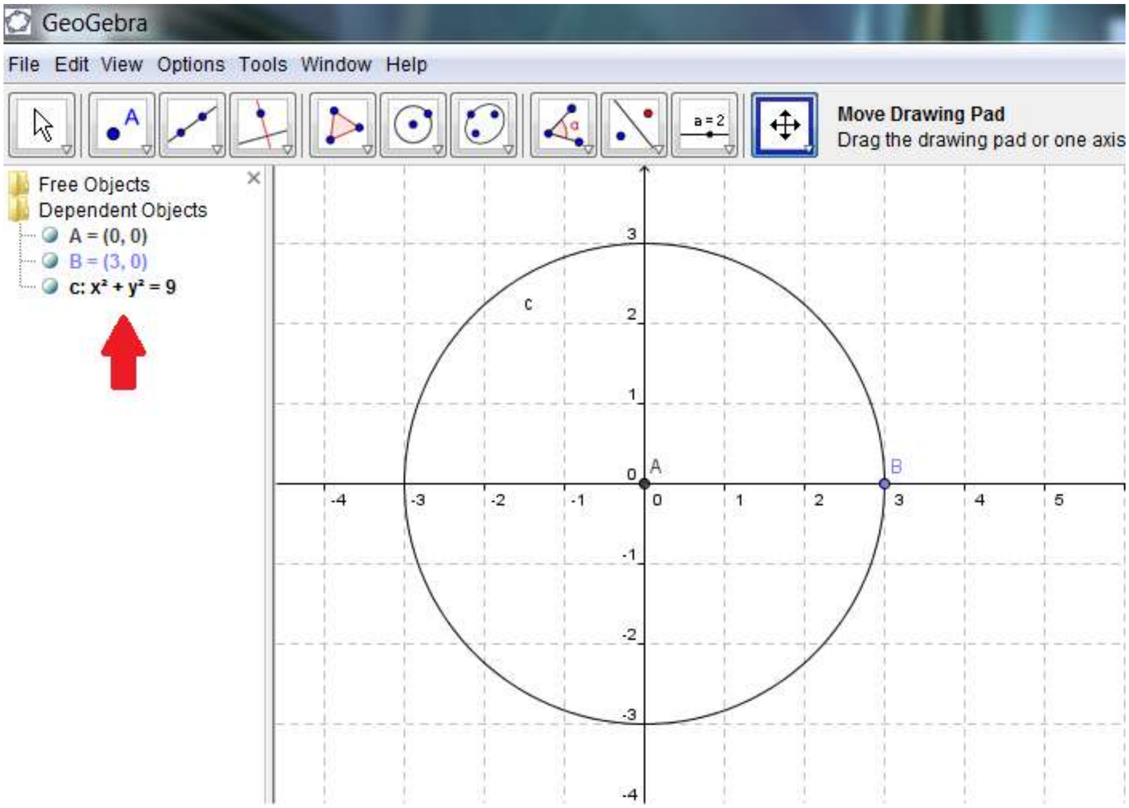
Materials

Geogebra

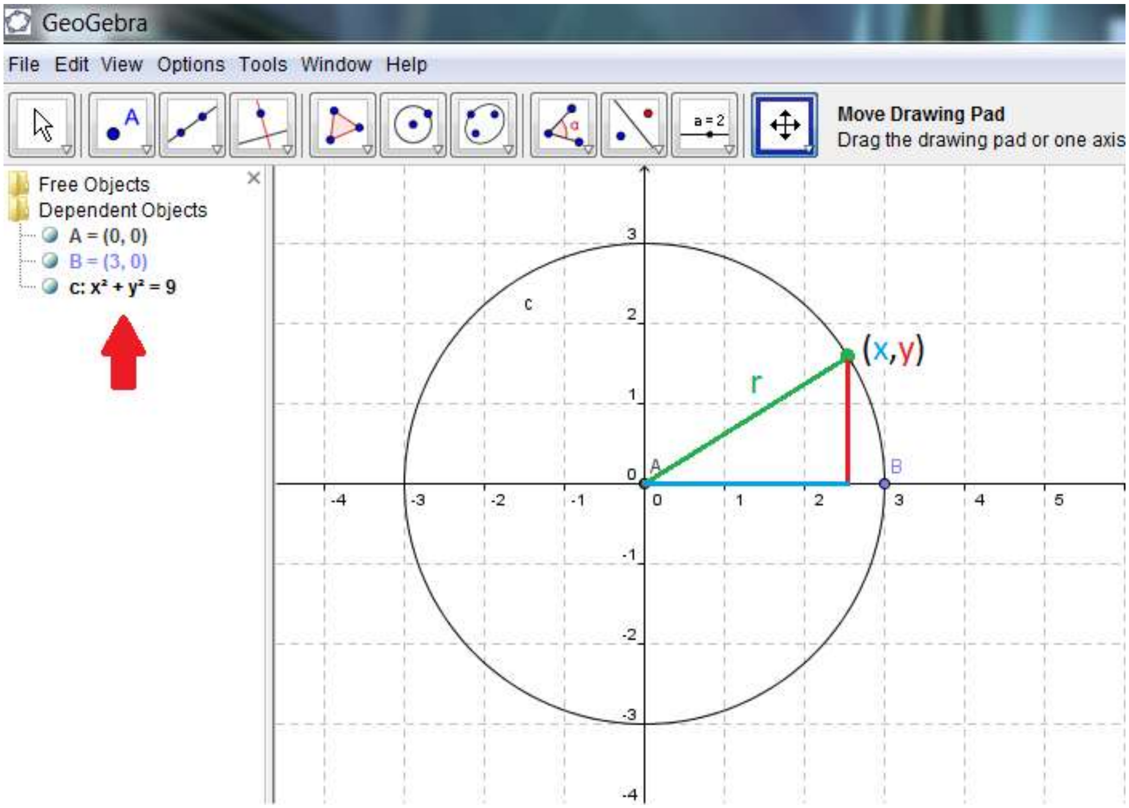
Procedure

When we used Geogebra before we always moved the axes out of the way so we can have a bigger drawing area. This time we need the intersection between the x-axis and the y-axis to be approximately in the center of the screen. Use button 11 to move the drawing pad. Right-click on the drawing area and select “Grid" from the menu that appears. A coordinate grid should show.

Now draw a circle with its origin at the center of the coordinate grid (0,0), and place point B on the number 3 on the x-axis (the horizontal axis) as shown in the picture below (image has been adjusted to fit your screen):



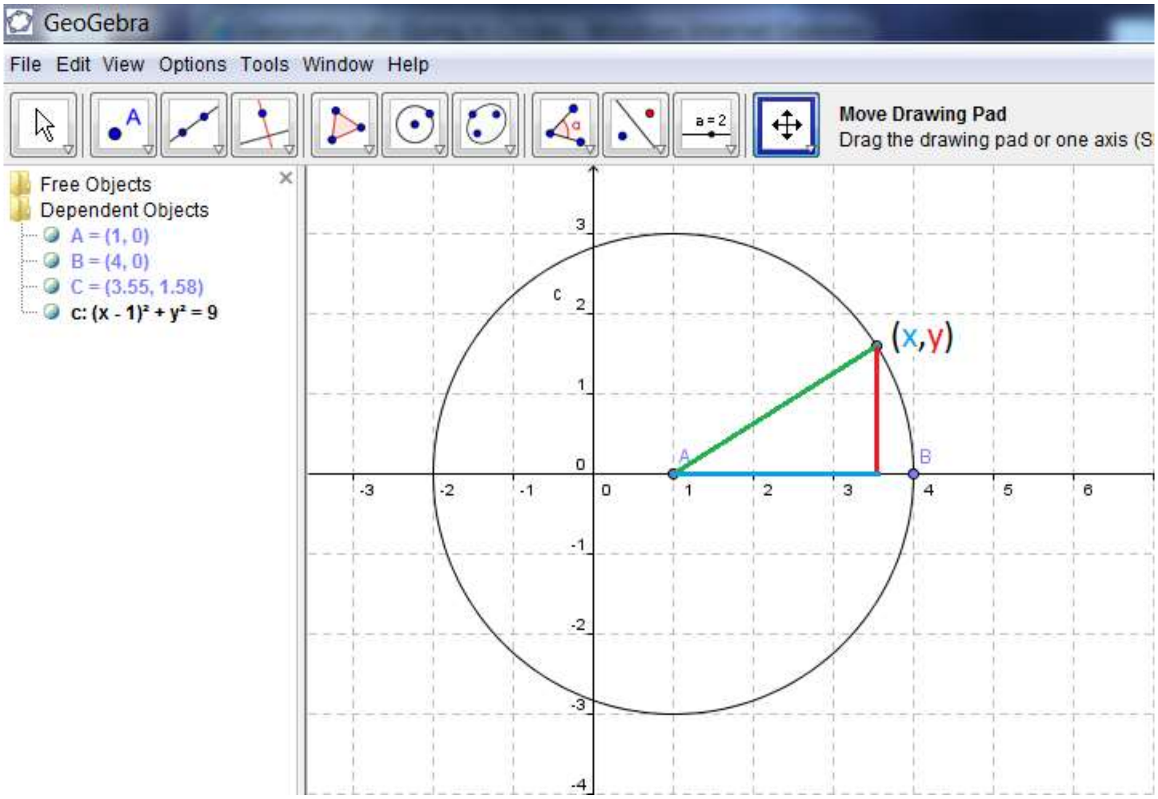
Notice that the equation of the circle appears automatically in the left-hand pane. I have marked it with a red arrow. (If for some reason the left-hand pane is not visible you can go to "View" at the top of your screen and select "Algebra View".) What does this equation mean? Well, for every value of x that you choose, there are two possible values of y so that the point (x,y) will be on the edge of the circle. The equation describes all of the points on the edge of the circle. For example for the point (-3,0), $x^2 = 9$ and $y^2 = 0$, so that $x^2 + y^2 = 9 + 0 = 9$.



In the picture above, you can see that I have marked a random point (x,y) on the edge of the circle. The blue line indicates the distance along the x axis, while the red line indicates the distance along the y, or vertical, axis. The green line labeled r is the radius of the circle. The Pythagorean Theorem tells us that $x^2 + y^2 = r^2$.

When you move point B on your circle to make it bigger, the radius changes. Try it out and look at the formula in the left-hand pane to see what happens mathematically.

It is not necessary to always put the center of a circle at the origin (0,0). Let's see what happens to the equation if you draw a new circle with the center at (0,1) and a radius of 3. Notice that now the equation looks a little different:



To find the x distance, which is the length of the blue line, you need to take the x coordinate, about 3.5 in this case, and subtract 1 to get a distance of 2.5. The length of the red line can still be found by taking the value of the y-coordinate. The formula becomes $(x - 1)^2 + y^2 = r^2$.

Analysis

What is the equation of a circle that has its center at (3,4)?

The general equation for a circle with its center at (h,k) is $(x - h)^2 + (y - k)^2 = r^2$.

Test 2

How to Balance a Line on a Circle

This is a very delicate procedure. A line that balances on a circle is a tangent line. The tangent line must touch the circle at exactly one point. Say that this is point A. The line should touch the circle at point A, but not at point B immediately next to it. If you consider that a point either has no size at all or is infinitely small in some way, you can see just how difficult this requirement is. After many failed tries with circular and straight objects I realized that it would be much easier to try this in reverse - we can balance the circle on the line.

Materials

Paper
Ruler
Construction Paper
Compass with sharp pencil point
Scissors
Random supporting object like a coffee mug or a can
Geogebra

Procedure

Draw a line on a regular piece of paper. This is the line that you will balance your circle on. Next, open your compass to a distance of about 6 cm. Draw a circle on construction paper, making sure to press the sharp point of your compass into the paper enough to leave a clear mark. Draw a line from this mark to the edge of the circle - this is a radius of the circle. Where that radius touches the edge of the circle will be point A. Carefully cut out your circle, staying exactly on the line drawn by your compass, especially near point A. The circle is very thin, so it will need some kind of support behind it to keep it from falling over as it balances on the line. I used a can of tomatoes laid on its side so I could lean the circle against the top of the can. A coffee mug with straight sides also works well - just lean the circle against the side of it.

Carefully balance the circle on the line so that it just touches the line at point A, the point of tangency. Then stand back - what do you notice about the angle between the line and the radius of the circle?

Repeat this experiment in Geogebra so that you can actually measure the angle very accurately. Tangent lines are found in the dropdown menu of button 4. Point A, or actually point B in Geogebra, can be anywhere along the circle.

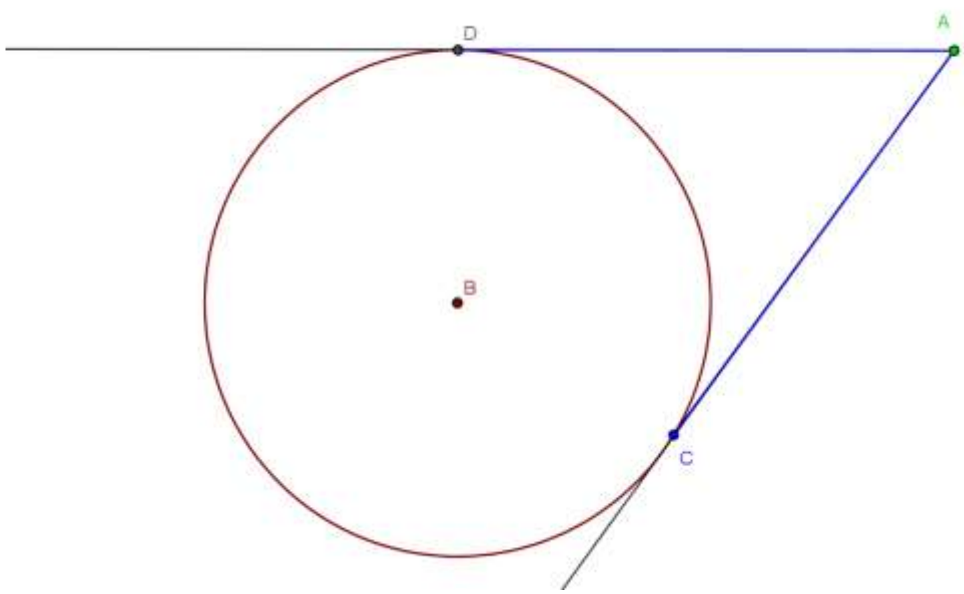
Analysis

Why should the tangent and the radius be perpendicular to each other? You can see it, but how would you prove it? After you have thought about it for a bit, read this clever proof by Euclid: http://aleph0.clarku.edu/~djoyce/java/elements/bookIII/propIII18.html. This is a proof by contradiction. You assume the opposite of what you are trying to prove, and then show that it could not possibly be true.

This proof makes use of the principle we learned in the experiment "Angles and Sides - Which One Goes Where?"

Practice

In the picture below, two lines have been drawn from point A tangent to the circle with center B. Show that segments AC and AD have the same length.



The Incenter: Putting a Circle inside Your Triangle

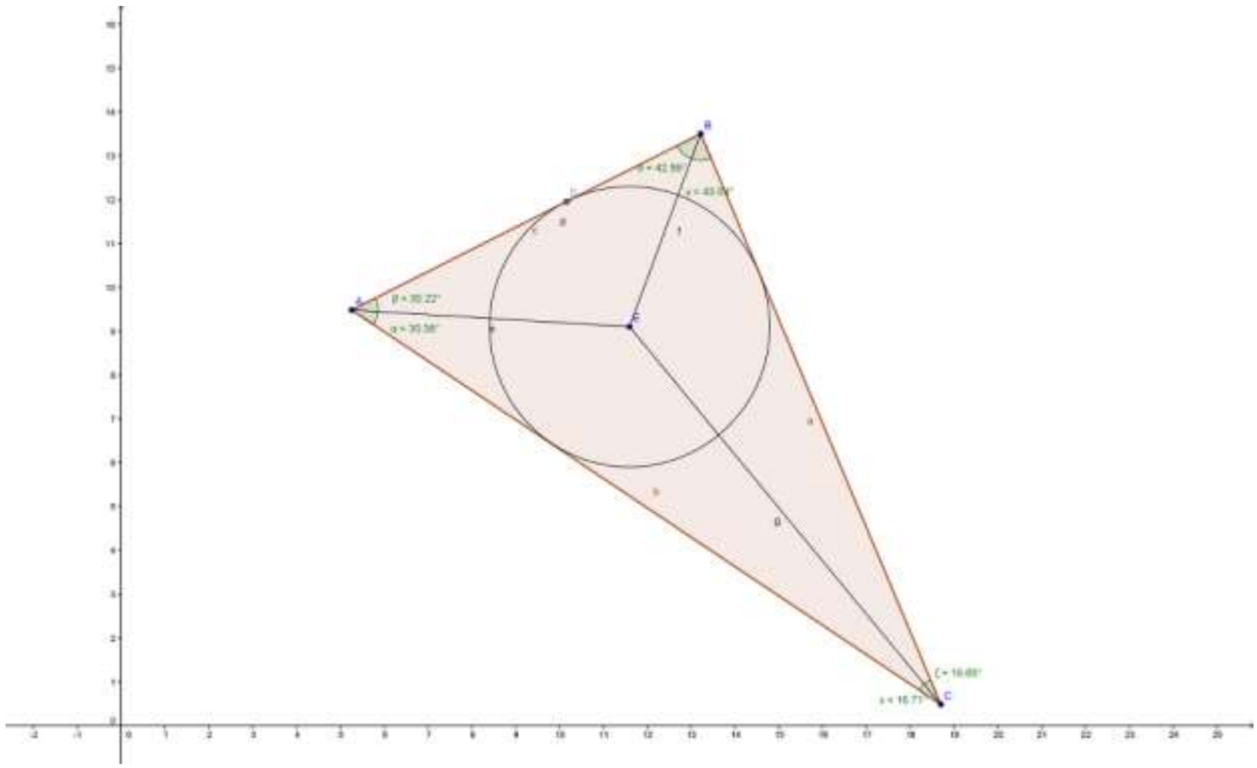
How big a circle can you fit inside a triangle? It depends on how hard you squeeze? Actually, you can put a small circle in your triangle, and then start making it bigger. When it gets big enough to touch one of the sides you can move it away from that side, and still keep making it bigger. What will finally stop you from continuing with this process? Will it be when the circle touches two of the sides? Or will it touch all three sides when it is as large as it can be? Make a prediction before you start up Geogebra to try it out.

Materials

Geogebra
Compass
Ruler
Paper

Procedure

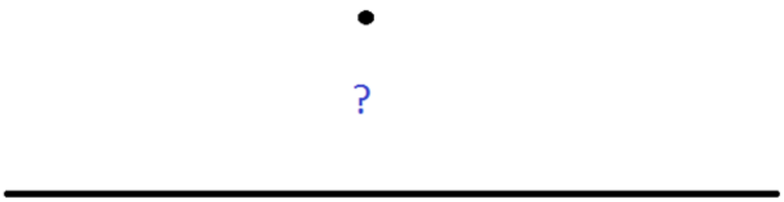
Draw a triangle in Geogebra, using the Polygon button. Draw a circle inside the triangle, and adjust it to be as big as possible while still keeping it completely inside the triangle. You may find it easier and slightly more accurate to place a point D on one of the sides of the triangle, and use this point as the marker for the circumference of the circle. Slide point D along the side of the triangle as you are adjusting your circle. As you work, you will see that it is not so easy to fit the circle into the triangle.

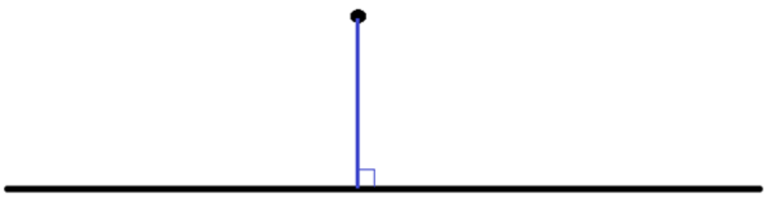


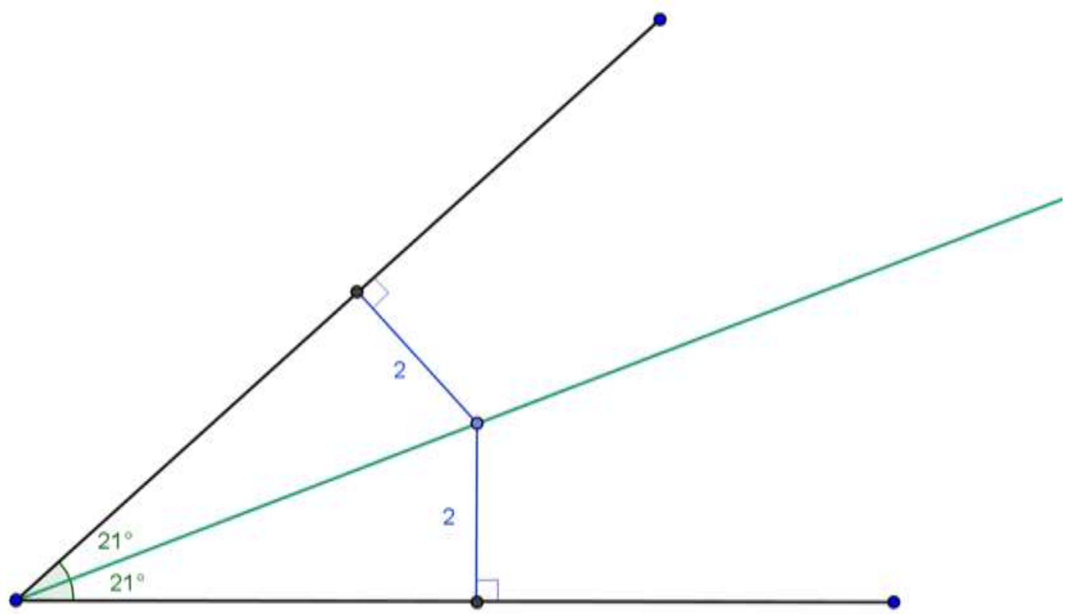
If we didn't have Geogebra to help us, we'd either have to do an awful lot of erasing or try to do the job a smarter way. It would be nice if we could know in advance just where to put the center of the circle. Take a closer look at where the center of the circle is in relation to the sides. The distance between each side and the center is equal to the radius of the circle, and that radius is the same length everywhere. Let's see if we can take advantage of that.

Grab a ruler and draw a random angle on a piece of paper. Use your compass to construct the bisector of the angle as seen here: http://www.mathopenref.com/constbisectangle.html

An angle bisector divides an angle into two equal parts. Pick a point on the bisector, and measure the distance between that point and the sides of the angle as shown in the picture below. The distance between a line and a point is measured by drawing a perpendicular line segment like this:

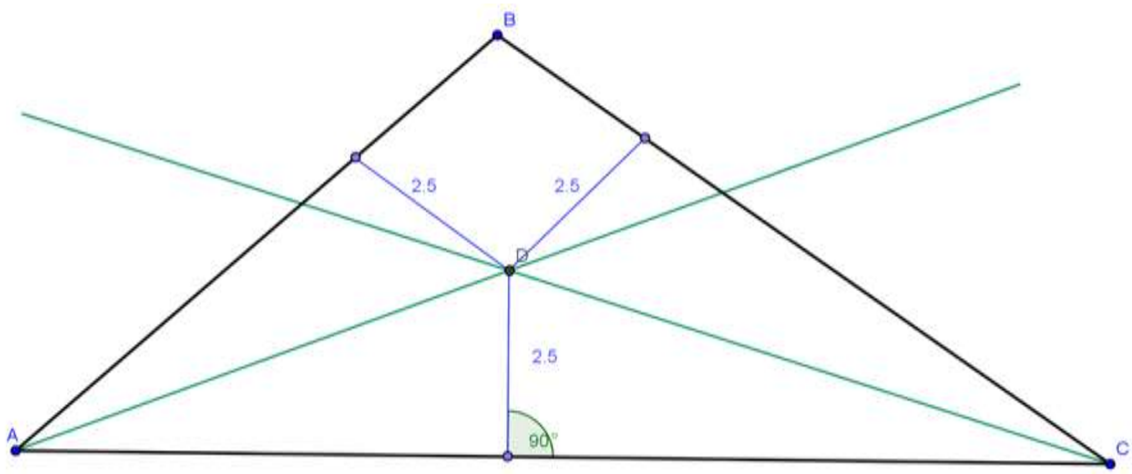






Any point that lies on an angle bisector ends up being **equidistant** (having an equal distance) from both of the sides that make up the angle. That's exactly what we need. Why do the blue segments marked "2" have to be equal in length? Look back at the experiment *Are Triangles With the Same Angles the Same?* if you are not sure why this is true.

Now draw a random triangle, and construct angle bisectors for two of the angles. At the point where those bisectors intersect, the distance to the edges of the triangle is the same in three places, because the blue segments are equal:



When you look at this picture you can see that now it is easy to draw a circle in the triangle. The center of the circle should be at point D, and the radius will be 2.5. Because the intersection of the angle bisectors of a triangle allows us to create an **inscribed** circle (a circle that just fits), this point is called the "**incenter**" of the triangle.

The picture above shows only two of the angle bisectors. Will the third angle bisector intersect the other two at point D? If so, why? Try it out by drawing your own triangle. If more than two lines intersect at a particular point, it is called a **point of concurrency**.

Analysis

It is not particularly surprising that a point on the angle bisector would be equidistant from both sides of the angle, since that makes a nice symmetrical picture. It actually happens because you end up drawing two congruent triangles using the point and the bisector. As you can see in the picture of the bisected angle, each triangle has a 90 degree angle and a 21 degree angle. They also have a common side, which is the angle bisector. That makes these triangles congruent by the AAS principle. The blue lines are equal in length because the two triangles are congruent.

The third angle bisector meets the other two at the point of concurrency, because there really is no other choice for it but to do so. Point D is equidistant from both sides of angle ABD, and therefore it has to be on the bisector of angle B.

Euclid explains how to inscribe a circle in a triangle here: http://aleph0.clarku.edu/~djoyce/java/elements/bookIV/propIV4.html.

The Circumcenter: Putting Your Triangle inside A Circle

Wait, didn't we just do that? Or no, that was the other way around. Anyway, is it possible to draw a circle around your triangle so that all three vertices of the triangle just touch the edges of the circle? Let's find out.

Materials

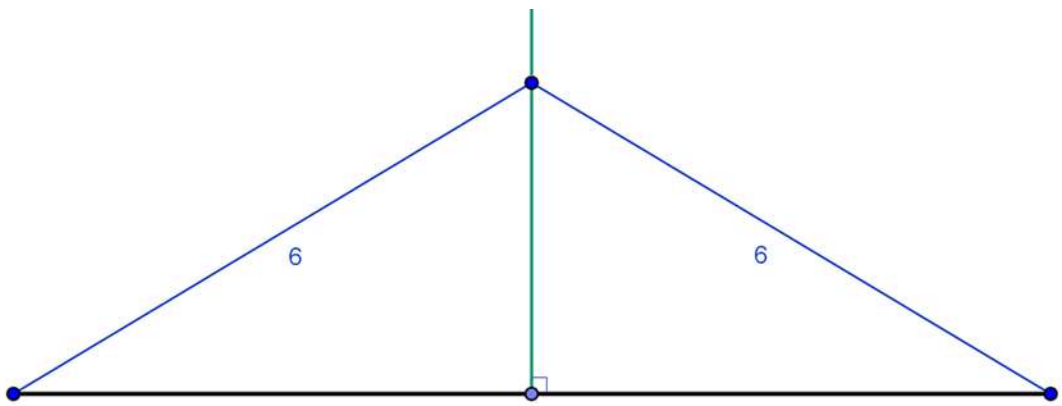
Geogebra
Compass
Ruler

Procedure

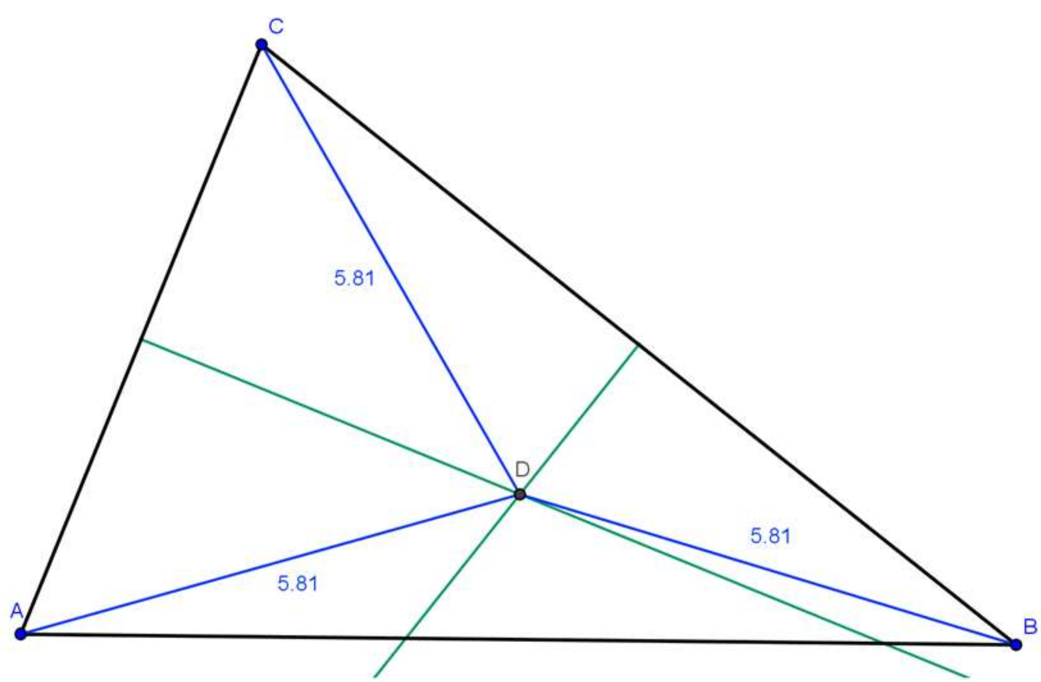
In Geogebra, create a triangle. Create a circle around the triangle. Move and resize your circle until all three vertices of the triangle are located on the edge of the circle.

The center of the circle that can be circumscribed (drawn around) the triangle is called the "circumcenter". If you did the last experiment you might suspect that there is a special way to lo locate this point, and in fact there is. Take a piece of paper and draw a line segment. Use your compass to create a perpendicular bisector of the segment, as shown here: http://www.mathopenref.com/constbisectline.html.

Now pick a random point on the perpendicular bisector, and measure the distance from that point to the endpoints of your segment. As you can see in the picture below, the two distances should be equal. Why is that?



Go back to your Geogebra drawing and construct perpendicular bisectors on two sides of the triangle. Use the perpendicular line button (button 4).



The point where the perpendicular bisectors intersect is equidistant from all three vertices of the triangle. This point is the **circumcenter**. The blue lines can be the radii of a circle, and with the center at point D the circle will touch all three vertices.

Check that the third perpendicular bisector actually intersects with the other two at the circumcenter. Do you think that the circumcenter would always be located inside the triangle? Change your triangle to see if you can position the circumcenter on an edge or even outside of the triangle. Can the circumcenter be outside of the triangle?

Analysis

If you draw a line segment and its perpendicular bisector, any point on that bisector will be the same distance from both endpoints of the segment. This is the **Perpendicular Bisector Theorem**. It works because you are actually drawing two congruent triangles. Both those triangles have one right angle and they share a side. The bases of your triangles are equal because you bisected the original segment. The triangles are congruent by the SAS principle.

In the picture above, the perpendicular bisectors of two sides intersect at the circumcenter, which is equidistant from the vertices of the triangle. The third perpendicular bisector will intersect the others at the circumcenter (point D), because that is the point where the distances to A and B are equal.

Read the proof supplied by Euclid: Euclid Book IV, proposition 5. Euclid shows that the lines extending from the circumcenter to each vertex are equal in length, because they are congruent parts of congruent right triangles. That means that if you put the point of your compass into the circumcenter and the pencil end at one vertex, you will draw a circle that has all of the vertices exactly on its edge. The triangle has been circumscribed by the circle.

Balancing a Triangle: The Centroid

It's easy to find the center point of a circle or a square.   It's right in the middle of the figure. But just where is "right in the middle" of a triangle?   Find out in this experiment!

Materials

Poster board
Compass
Ruler with centimeter markings
Mechanical pencil or very sharp regular pencil

Procedure

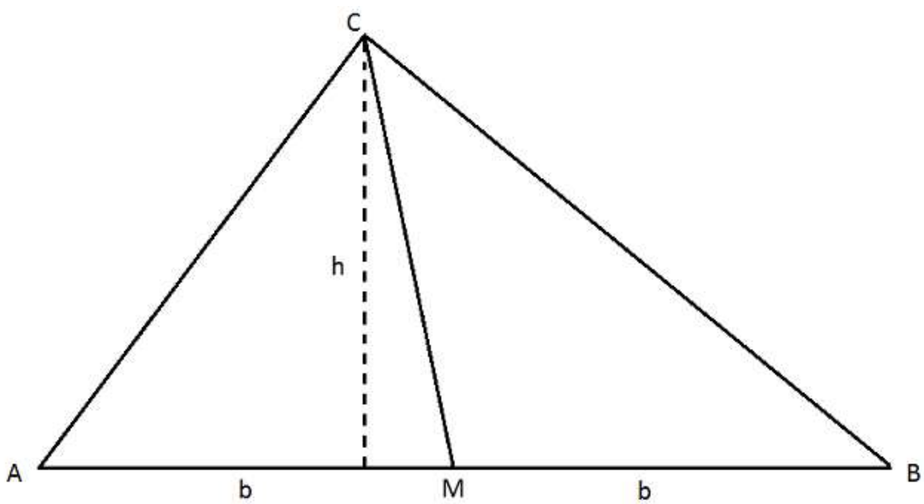
For this experiment to be successful, you must work very carefully and precisely. Since we are looking for the exact middle of a triangle every millimeter is important. We need a sharp pencil so that the position of vertices we draw or any marks that we make is exact.

Draw a random large triangle (no bigger than a standard letter-sized piece of paper) on poster board. Find the exact middle of one of the sides, and carefully mark it. Then draw a line from this mark to the opposite vertex of the triangle. This kind of line is called a **median** of the triangle. Repeat with the other sides.   If you did it just right, all of the lines will intersect in one single point: the centroid of the triangle. Now cut out your triangle very carefully. Make sure you cut exactly on the lines.

To see if this is really the "center" of the triangle, we will try to balance the triangle at this point.   Open your compass wide and carefully insert the point into the centroid without actually going through to the other side.   Lift the triangle up and see if it balances on the point of your compass.   How close did you get?   This balance test is extremely sensitive.   If your triangle is not straight try carefully moving the point of your compass to a slightly different spot.   Is it straighter now?   Next, try moving the point of your compass to a spot just a few millimeters away from the centroid.   Does the triangle balance anywhere other than very close to the centroid?

Analysis

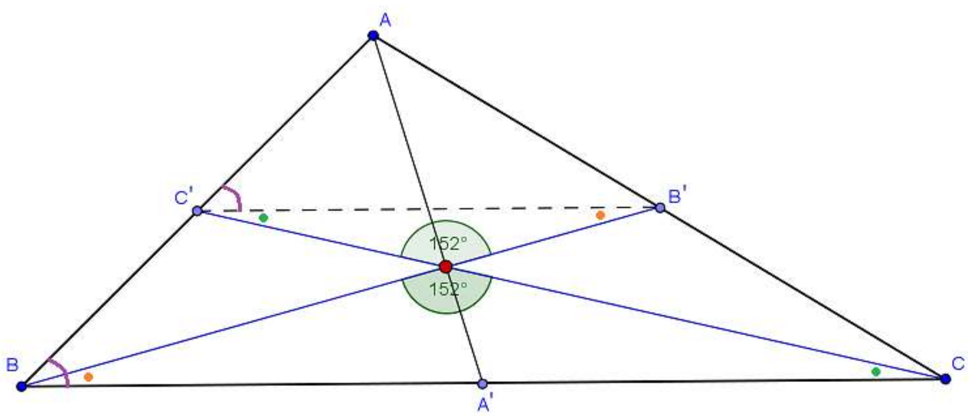
For the triangle to balance, the point of the compass has to be in the center of the area of the triangle, so that equal amounts of poster board are pulled down by gravity on all sides. Whenever you draw a median of a triangle, that median divides the triangle into two equal parts.



Here the original triangle, ABC, has height h and base 2b. Its area is ½ times 2b times h. That multiplies to bh. The median CM divides triangle ABC into two smaller triangles. Triangle AMC has base b and height h, so its area is ½ b times h, or ½ bh; exactly half of the area of triangle ABC.

If you draw all three medians, you divide the triangle up into 6 areas that are all equal in size, and therefore in weight. The triangle balances at the centroid.

Interestingly, the centroid can be found along any single median by measuring out 2/3 of the distance from the vertex to the opposite side. In the picture below, the distance from point B to the centroid is exactly 2/3 of the total distance from point B to point B':



Is that magic? Of course not! Look carefully at the picture above. Medians extend to the midpoint of the opposite sides, so C' and B' are located exactly in the middle of each side. Triangles AC'B' and ABC are similar triangles because they share a common angle at A and their sides are proportional (SAS similarity). The proportion here is 1:2 – every part of triangle ABC is twice as big as that of AC' B'. In particular, BC is twice as long as B'C'.

Similar triangles have identical angles, so the corresponding angles AC'B' and ABC (marked with little purple curves) ensure that C'B' is parallel to BC. Parallel lines then create the Z-angles marked with colored dots.

The two angles labeled 152 degrees are equal because they are vertical angles. Notice that the triangle formed by B, C and the centroid has the same angles as the triangle formed by B', C' and the centroid. These two triangles are similar. Since BC is twice as long as B'C' the proportion is 1:2. It is this proportion that causes the centroid to be 2/3 of the distance along the median. The centroid divides the median into two parts, and one is exactly twice as long as the other.

Finding Your Center: The Orthocenter

The orthocenter is the last type of triangle center that we will cover in our labs. Although there is nothing particularly exciting about the orthocenter itself, it is quite popular among educators because it provides an opportunity to have students review equations and intersections of lines. Many students find these topics somewhat difficult, so here are the relevant reviews. If you are taking geometry before you have taken algebra (a more historically correct approach) you can skip this section.

Materials

Graph Paper (paper with little squares instead of lines)
Ruler
Protractor
Geogebra

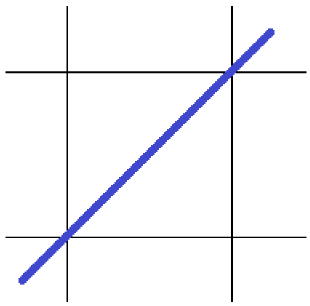
1. Coordinate and Slope Review

When you are working with coordinates, remember that the first number is the distance along the x-axis (the horizontal numbered line), and the second number represents the distance along the y-axis (the vertical numbered line). If you need more review, watch the movie at http://www.youtube.com/watch?v=HdrCwFNcXGU&feature=related

The slope of a line is $\frac{\text{Rise}}{\text{Run}}$. If that doesn't look familiar you can do a quick review now.

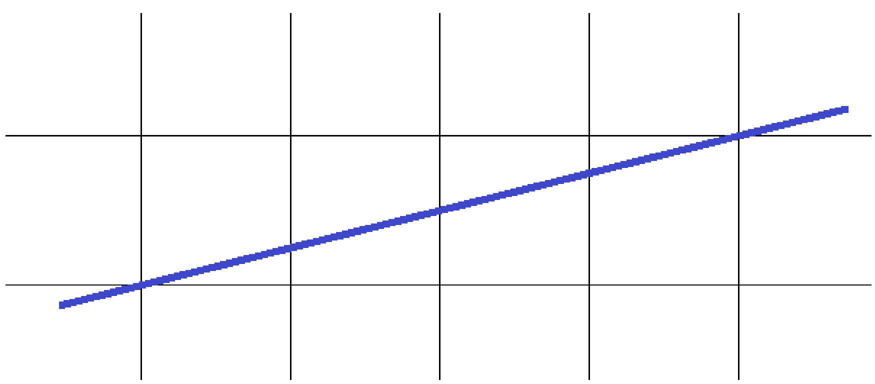
Procedure

Take a sheet of graph paper, and using a ruler, draw a line from the lower left corner of a square to the upper right corner, like this:



Two points define a line. Extend the line carefully, and you will see that it passes exactly through the corners of every square it crosses.

If this line represented a hill, that hill would be very steep. Walking up this hill would be difficult, and riding a bike could be dangerous. Let's create a line that is less "steep". Select 4 adjacent horizontal squares on your paper, a little below your first line. Now carefully draw a line from the left lower corner of the square that is furthest to the left, to the top right corner of the rightmost square. Extend your line through the next group of 4 squares and so on.



If you imagine that this line is a hill, you'll notice that it is not nearly as steep. People have developed ways to measure almost everything, and the steepness of hills is no exception. The standard way to measure the slope of a hill is called "**rise over run**", which means you look at the amount the hill goes up over the horizontal distance that you are measuring. Our first line went up 1 square vertically for every square we moved to the right horizontally. The slope of this line is 1 / 1 which is 1. The second line goes up only 1 square for every 4 horizontal squares. The "rise" is 1 and the "run" is 4, so the slope is 1/4 or 0.25. Notice that it makes no difference if we measure over a longer distance, such as 8 squares horizontally. The rise is 2 squares per 8 horizontal squares, so the slope is 2/8 which is still 0.25.

In a Cartesian coordinate system, not only do two points define a line, they also define its slope. Draw x and y-axes on your paper. Create a line segment from the point (0,1) to the point (6,4). You can extend this line in both directions in only one way, and it always stays at the same angle relative to the x-axis and the y-axis. If you've drawn your line carefully, you can measure its slope anywhere, but let's measure it between the two points I mentioned. Start by putting your pencil at (0,1). To get the "run" or horizontal part, move the point of your pencil 6 units to the right. Because the x-axis is a number line, we are moving in a positive direction by going to the right. For the "rise", or vertical part, notice that your pencil is at 1 relative to the y-axis. Move up to get to 4, which is 3 units. Again notice that we are moving in a positive direction to get the rise. What is the value of the slope? You can check your answer by drawing the same line in Geogebra. Use the dropdown menu on button 8 to mark the slope.

[If you have trouble seeing where the rise and run are, check out the applet at http://www.mathwarehouse.com/algebra/linear_equation/interactive-slope.php]

But what if we start at the other point? Put your pencil at (6,4). Now, to get the "run" you have to move in a negative direction: 6 units to the left. This means that the run is now -6. And the rise.... seems to have turned into a drop. You need to move your pencil down 3 units to get to (0,1). The rise is -3. You should get the same value for the slope as you did before. This means that if you are calculating the slope of a line from two given points, it doesn't matter which point you start at, which makes sense since the slope should be the same either way.

It is also possible to have a line with a negative slope. To see one, use the points (0,8) and (2,0) to draw a line. Starting at (0,8) the run is 2 units in a positive direction. The rise is a drop, or -8 units. What is the slope of this line? Check that the slope works out the same if you start at the other point.

If you create a horizontal line in Geogebra you can see that is has a slope of zero, since the rise is 0. If the run is 0, well, we can’t divide by zero so the slope of a vertical line is undefined.

2. Parallel Lines

Procedure

Draw a line with the equation y = 2x. To do this, pick some random value for x, and then calculate the value for y. For example, if x is 1, y would be 2. If x is -2, y will be -4, and so on. To make sure you have the right line, compare the one on your paper to the same line drawn in Geogebra. Geogebra lets you type the equation into its Input field at the bottom, and then draws the line for you immediately.

Next, measure the slope of your line and record it. Compare your value for the slope with the one Geogebra gets (Slope is available from the dropdown menu on button 8). Notice that Geogebra uses the letter m for the slope.

On paper, draw and record the slopes of three more random simple lines, such as for example y = 3x, y = -x, or y = $\frac{1}{4}$ x. Check your work in Geogebra (y = $\frac{1}{4}$ x should be entered as y = (1/4)x or y = 0.25x). How is the equation of the line related to the slope? In what way do lines with a negative slope look different from lines with a positive slope? Draw more lines as needed to see the difference.

Next, draw the following lines:

y = 2x + 1
y = 2x + 3
y = 2x – 4

What does adding or subtracting a number at the end do to the line? These lines are parallel. What do parallel lines have in common?

3. Perpendicular Lines

Procedure

Create a coordinate system on graph paper, and draw a line through the points (0,0) and (8,4). Determine the slope of this line. Next, take your protractor and very carefully create a perpendicular line through the point (8,4). Measure the slope of this perpendicular line.

Multiply the slope of the original line by the slope of the perpendicular line. What is the resulting number?

Create two points in Geogebra and draw a random line through them. Create a new line perpendicular to this line through one of the two points (button 4). Mark the slope of both lines. Multiply the two slopes and record the result. Move your points to a different location and multiply the slopes again. Do you think the result will always be the same?

4. Creating Lines from Equations

The general equation of a line is y = mx + b, where m and b can be positive or negative numbers, or 0. A horizontal line has a slope of 0, so its equation is y = 0x + b, or y = b. The equation of a vertical line is created by specifying a value for x, such as x = 3. This will create a vertical line for which all of the points have an x-coordinate of 3. Try it out in Geogebra.

Procedure

In Geogebra, create two points – one at (2, 3) and the other at (4, 9). Create a line through these two points. First calculate, and then mark the slope of the line. Look at the equation of the line in the Algebra section on the left. If the equation does not appear in the standard y = mx + b format, just right-click on it and select Equation y = mx + b. Part of the equation is predictable, because the value of the slope always appears in front of the x, in place of m. Do you think there is any way to predict the value of b?

Copy the equation of the line on paper. The point (2, 3) is a point on the line, so it should fit the equation. Rewrite the equation using 3 as the y-value and 2 as the x-value. Is the equation true? Also check the point (4, 9) to see if it fits the equation.

Next, you will create the equation of a line with a slope of -2 that passes through the point (1, 5). Since the slope of the line is -2 you can use that as the value of m. Although the value of b is unknown, you can choose it so that the point (1, 5) is on the line. Use your algebra skills to find b, or just guess until you get it right. Once you think you have the correct equation, enter it into Geogebra to see if you were right.

Analysis

Suppose we need the slope of a line to be -3, and have it pass through the point (2, 7). The equation of the line is $y = mx + b$, or $y = -3x + b$ because the slope m is -3. This equation has to work for any point on the line, so we can temporarily insert $y = 7$ and $x = 2$:

$y = -3x + b$

$7 = -3(2) + b$

$7 = -6 + b$ Add 6 to both sides:

$13 = b$

Now we know the value of b. Put that value into the general equation $y = -3x + b$, so we get $y = -3x + 13$.

Many students prefer to use the point-slope form for creating these equations. This form can be created through algebra by solving a system of equations. Since $y = mx + b$, it is also true that $y_1 = mx_1 + b$, where (x_1, y_1) is a point on the line. Solve by subtraction to eliminate b:

$y = mx + b$
$\underline{y_1 = mx_1 + b}$
$y - y_1 = mx - mx_1$

Factor out m to get $y - y_1 = m(x - x_1)$. This form is nice because you can just plug in a slope and a point to get what you want. For a line with a slope of -3 passing through the point (2, 7):

$y - y_1 = m(x - x_1)$

$y - 7 = -3(x - 2)$

Use the distributive property to change that to:

$y - 7 = -3x + 6$

Add 7 to both sides:

$y = -3x + 13$.

5. Intersecting Lines

Procedure

Create a point with coordinates (5,3) in Geogebra. Place two additional points so that you can create two random lines that intersect at the point (5,3). Look at the equations of your lines in the Algebra View on the left. If they do not display in the $y = mx + b$ format, right-click on them to change that. Now adjust your lines until both equations contain only whole numbers. My equations were $y = x - 2$ and $y = -x + 8$. There are many different values for x and y that fit into these equations. However, at the intersect point the x and y values are the same for both equations. At this point, y is equal to $x - 2$, and also equal to $-x + 8$. That means that $x - 2$ and $-x + 8$ both have the same value, so we can set them equal to each other:

$x - 2 = -x + 8$

Add x to both sides to get

$2x - 2 = 8$

Then add 2 to both sides:

$2x = 10$

Divide both sides by 2:

$x = 5$

This is the x-coordinate of the intersect point, as we already knew. To get the y-coordinate, we can use the equation of either line since both lines have the same y value when x is equal to 5. For $y = x - 2$ we get $y = 5 - 2$, so y must be 3. (5, 3) is the calculated intersect point. To be sure that you didn't make any mistakes, you should try these values in both equations. For $y = -x + 8$, it is true that $3 = -5 + 8$.

Practice with some different intersect points until you can easily find the intersection of two lines from their equations. Always check your answer in both equations.

6. Finding the Orthocenter

The orthocenter of a triangle is the point where the altitudes intersect. The altitude is the "height" of a triangle. It is a line from the "top" vertex straight down to the bottom, so that it makes a 90 degree angle with the bottom edge. If you turn your triangle so that another vertex is at the top, you can draw a different altitude. Because the altitudes are lines from a vertex (which is a point) perpendicular to the opposite side, it is possible to find the equation of each altitude. To do this you must construct a line perpendicular to a side that also passes through the opposite vertex. You will need the slope of the side, and the coordinates of the opposite vertex. Once you have two altitudes you can determine where they intersect. The third altitude should intersect the other two at the same point.

Procedure

Create triangle ABC on paper or in Geogebra by placing three points at the following coordinates:

Point A at (2,1)
Point B at (16,1)
Point C at (5,12)

If you placed these points in the proper order, the top of the triangle is at point C, and segment AB makes up the base of the triangle. Find the equation of the line that represents the altitude from point C to the opposite side, AB. This is a vertical line, so it can be specified by using a value for x. Then find the equation of the line that represents the altitude from point A to the opposite side BC. This will be sufficient to locate the orthocenter of this triangle. The third altitude is more difficult to find, but you can see if you can do it so that you can verify that it intersects the other two lines at the calculated orthocenter. What are the coordinates of the orthocenter? Show all of your work. Check that you have the right answer by using Geogebra. Once you have done so, change the shape of the triangle to see if you can position the orthocenter on an edge or even outside of the triangle. Can the orthocenter be outside of the triangle?

Triangle Centers : Summary

Name	Concurrency of	What it does	Location
Incenter	Angle bisectors	Puts a circle in the triangle	Always inside the triangle
Circumcenter	Perpendicular bisectors	Puts a circle around the triangle	Inside acute triangle, outside obtuse triangle, on midpoint of hypotenuse of right triangle
Centroid	Medians	Indicates the balance point, divides medians in a 2 to 1 ratio	Always inside
Orthocenter	Altitudes	Creates a problem for you to solve	Inside acute triangle, outside obtuse triangle, on vertex of right triangle

In an equilateral triangle all of these centers are in the same place.

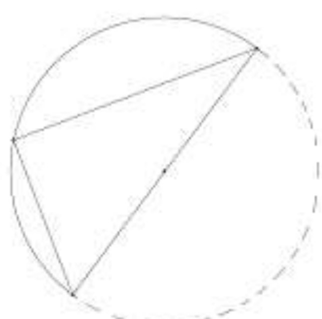
An Angle in a Semicircle

Materials

Protractor
Ruler
Paper
Geogebra
Compass

Procedure

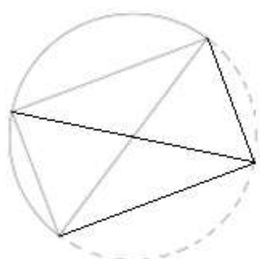
On paper, draw a semicircle and draw a triangle in it as shown. Two of the vertices of the triangle are the endpoints of the diameter of the circle, and the third vertex is on the edge of the semicircle. The triangle is *inscribed* in the semicircle. See if you can create an inscribed triangle in a semicircle that is **not** a right triangle. Report your findings.



Draw a circle in Geogebra. Point A will be the center of the circle, and point B will be on the edge. Place point C so that it is roughly opposite point B, and create segment BC. Mark the length of segment BC. Under Options → Rounding, set rounding to 5 decimals places. Move point C until segment BC is a diameter of the circle. This will occur where segment BC has the longest possible length. Place point D on the circle so you can inscribe triangle BCD. Move point D around. Does the angle change at all?

Analysis

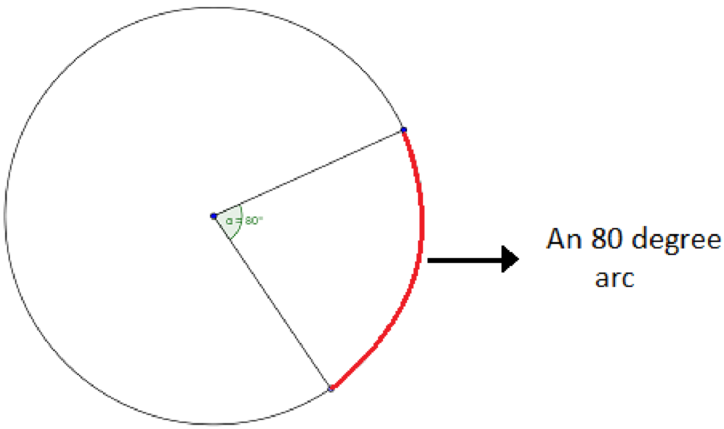
Using the triangle and the semicircle you drew on paper, complete the circle. Draw a line from the vertex of the inscribed angle through the center of the circle to the opposite side. This line will also be a diameter of the circle. Complete another triangle as shown:



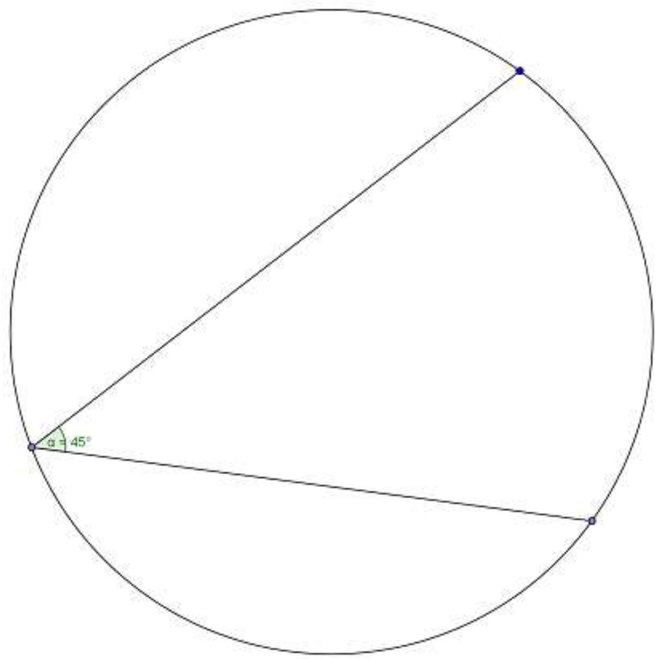
There are now 4 smaller triangles inside the circle. What kind of triangles are they? Mark all of the angles that you can prove are the same. Explain why a triangle inscribed in a semicircle must be a right triangle. Caution: do not make assumptions that you cannot prove.

An Angle in a Circle

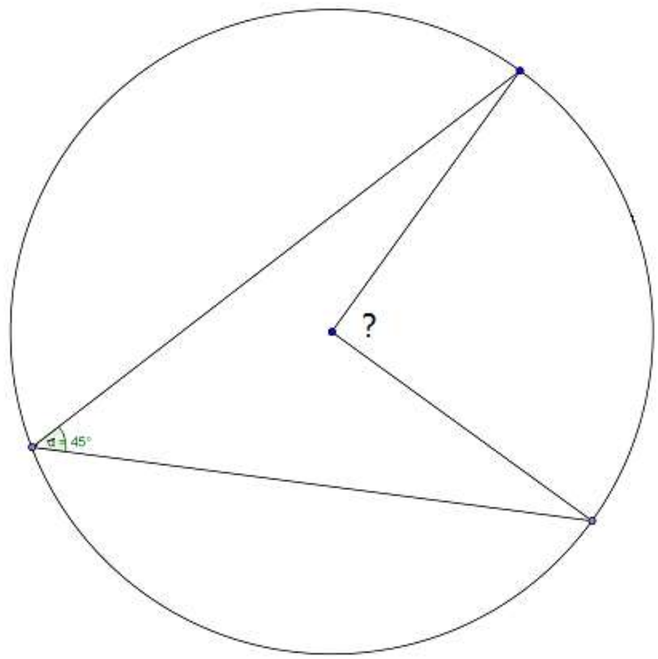
The previous experiment was actually a special case of the more general idea that we will explore in this experiment.



Here is an angle inside a circle. Because the vertex or point of the angle is at the center, we call this a central angle. This angle measures 80 degrees. A circle is divided into 360 degrees. Just as this angle is a part of the total circle, it also measures out an **arc** or curved line that is a part of the total circumference of the circle. We say that the total circumference of the circle is also 360 degrees, so this arc is 80 degrees. The measure of a central angle is always equal to the arc it intercepts. That is rather straightforward. Now let's look at another kind of angle:



This is called an inscribed angle, because it is "inscribed" in the circle. All three points that determine the angle are located on the edge of the circle. Wherever there is room to ask a question, someone eventually will, and this situation is no exception. Some inquisitive person who lived a very long time ago said something like, "I see that the inscribed angle is 45 degrees, but what is the measure of the arc that it intercepts?" Since we just saw that the measure of the arc is the same as the measure of the central angle, we can represent the question this way:



As it turns out, this is actually a rather interesting question with a rather surprising answer.

Materials

Protractor
Ruler
Paper
Geogebra
Compass

Procedure

Draw a large circle on paper. Inscribe an angle inside the circle as shown above. Measure the angle and the corresponding central angle. Record your measurements.

Draw a large circle using the circle button in Geogebra. Create an inscribed angle by placing two new points on the edge of the circle (do not use point B because it controls the size of the circle). Add line segments so you can see the corresponding central angle. Click on the angle button and then on the line segments enclosing the angle to display angle measurements.

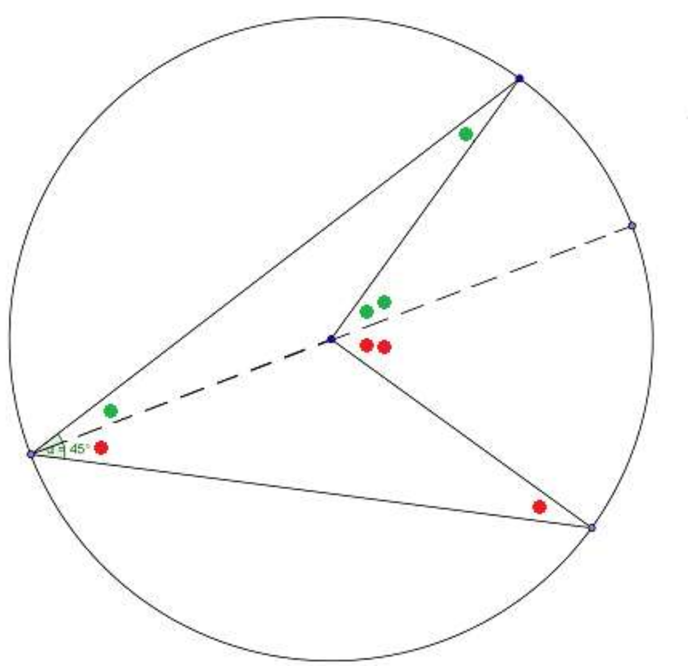
Record the measurements of both angles. Use the move button to drag the numbers to a better spot if you have trouble reading them.

Move the points of your angle around so you get a different inscribed angle. Record the new measurements. Repeat this twice so that you have measurements for 4 inscribed angles altogether.

Analysis

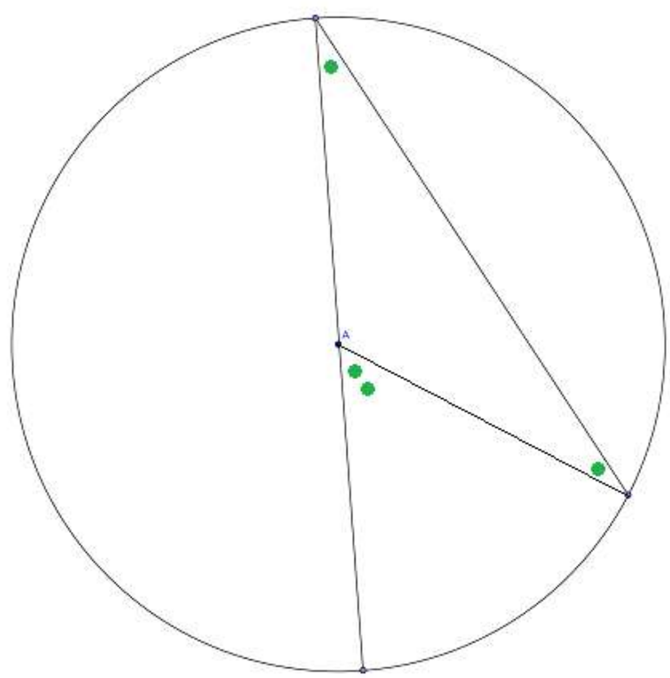
What do you notice about the relationship between an inscribed angle and its corresponding central angle?

Euclid explains this finding very nicely in his third book (Book 3, proposition 20). Because it can be a bit of a chore to figure out which angles he is talking about, you may want to look at this slightly more detailed version of his proof that uses colored dots for the angles. Notice that an extra line has been drawn, creating two triangles. These are isosceles triangles because two of their sides are radii of the circle. Single colored dots have been placed where the angles of the isosceles triangles are equal.

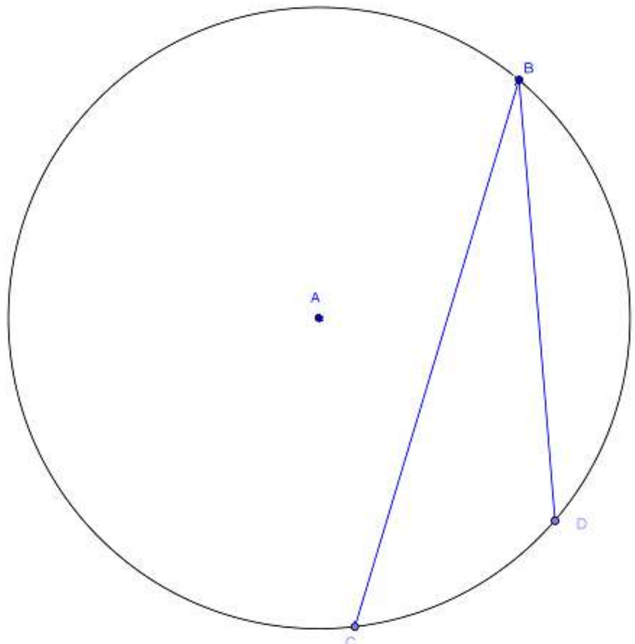


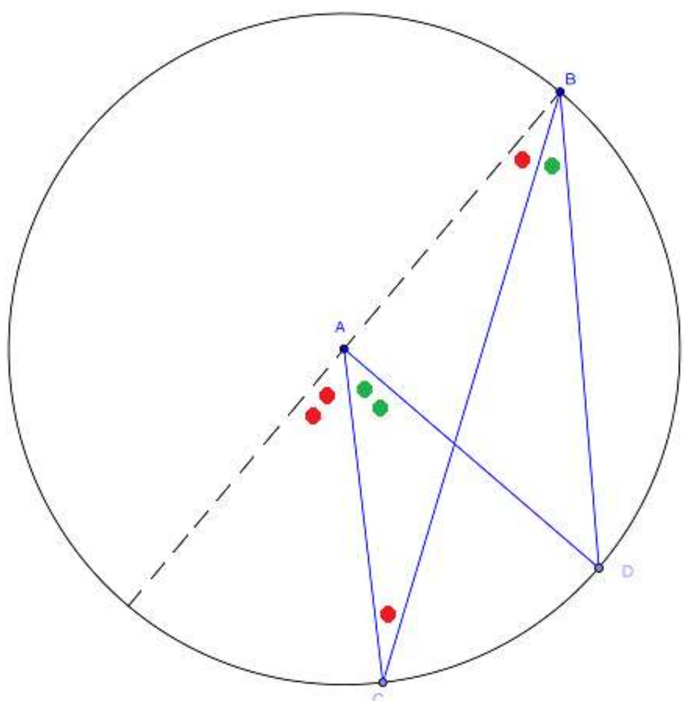
Notice that inside each triangle, the unmarked angle plus the two angles marked with dots add up to 180 degrees. The unmarked angle plus the angle with the two dots also add up to 180 degrees because these are supplementary angles (angles that make a straight line). As a result, we can be sure that the angles marked with two red or two green dots are twice as big as the angles marked with one red dot or one green dot. Our original 45 degree angle is made up of an angle with a green dot and an angle with a red dot. The corresponding central angle has two green dots and two red dots, so it must be twice as big.

When the inscribed angle has a diameter of the circle as one side, it is even easier to see what is happening with the central angle:



Euclid also addresses this situation, which is a little trickier:


B
A
D
C

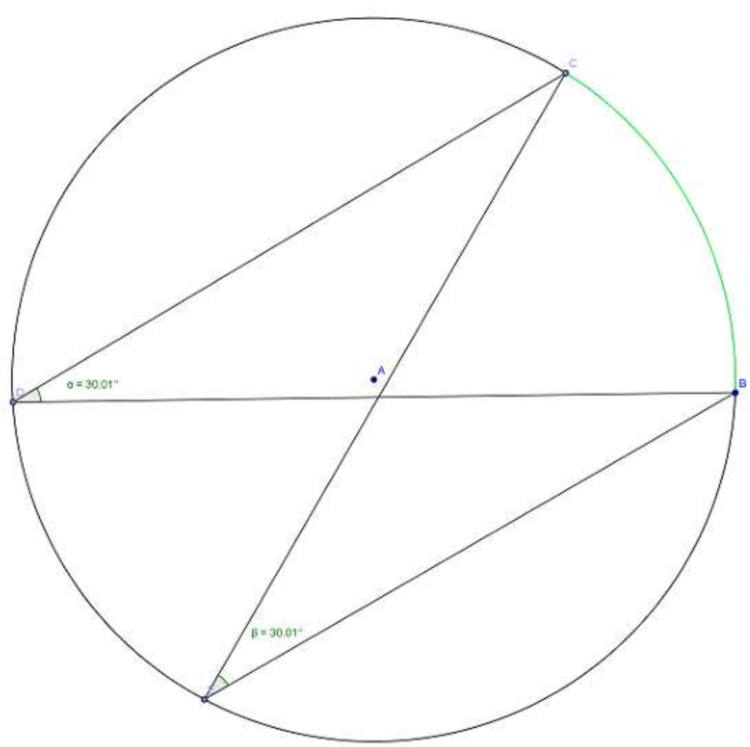

B
A
D
C

We can prove the arrangement of the red dots, as we did before. Next, we place a single green dot in the original inscribed angle at B. Notice that the angle that is the sum of the green and red dots is really another inscribed angle that has the diameter of the circle as one of its sides. Its corresponding central angle should be twice as large, and can be marked with two red dots (that were already there) plus two green dots. Now we can see that the angle marked by the green dot is exactly half of the angle marked by the two green dots, so it is half the size of its corresponding central angle and therefore half the size of the arc that it intercepts.

http://aleph0.clarku.edu/~djoyce/java/elements/bookIII/propIII20.html

An angle inscribed in a semicircle intercepts an arc of 180 degrees. As a result, the inscribed angle must be 90 degrees as we saw in the last experiment.

Now draw the following picture in Geogebra:



What do you notice about the two marked angles? What happens to these angles as you move the points on the edge of the circle? This is caused by the fact that these two inscribed angles have their endpoints on the exact same arc!

Now that you understand that inscribed angles are half the measure of the arc they intercept, you can use that to understand the following interesting fact:

"If you draw a random 4-sided polygon inside a circle so that all of the vertices are on the edge of the circle, its opposite angles will sum to 180 degrees." Check it out here: http://www.geom.uiuc.edu/~dwiggins/conj47.html

Write your own explanation that explains why a quadrilateral inscribed in a circle must have opposite angles that add to 180 degrees.

A Chord and a Tangent

A chord is any line segment that has its endpoints on the edges of a circle. A diameter of a circle is also a chord.

A tangent to a circle is a line that touches the circle at only one point.

Materials

Protractor
Ruler
Paper
Geogebra
Compass

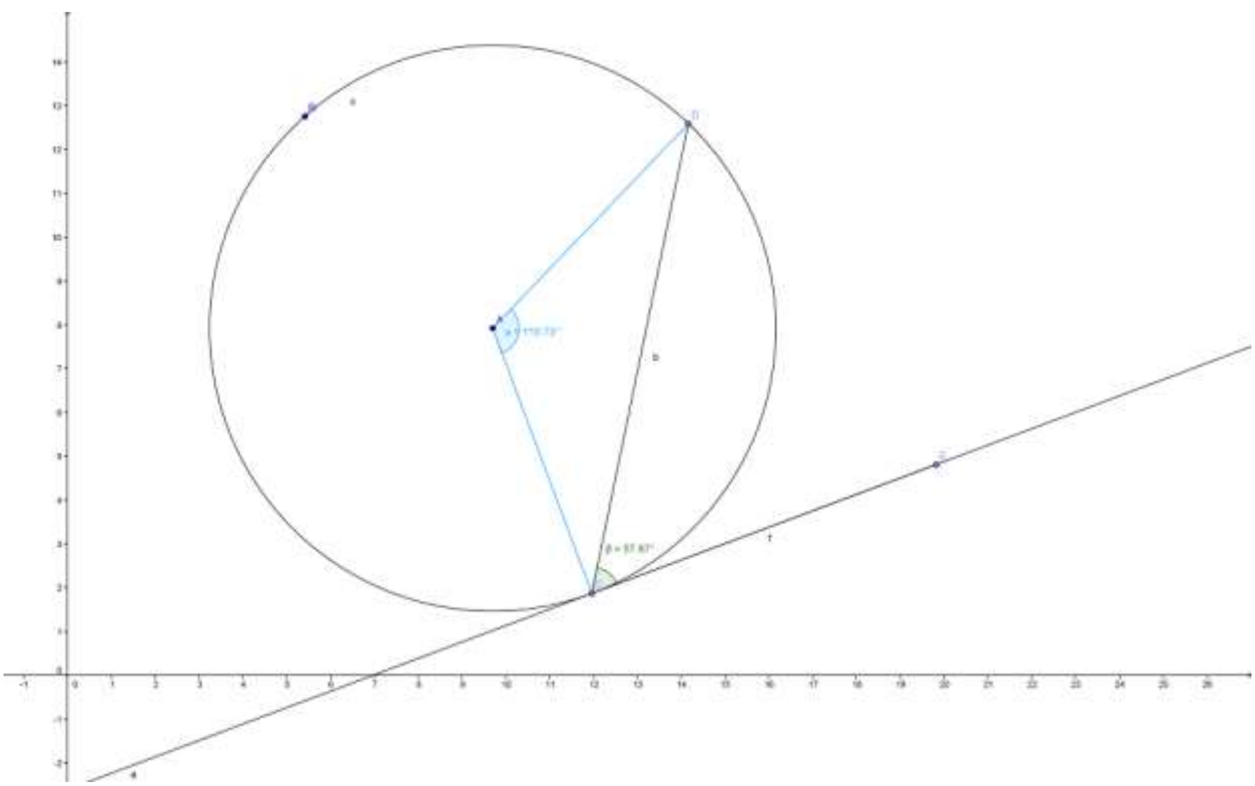
Procedure

Draw a circle, mark a point P on the edge, and try to draw a tangent at that point. You can see that you have to place your line just exactly right, so that the curve of the circle moves away from the line equally in both directions. There is a simple way to accomplish this. Draw a line segment from point P to the center of the circle. Using a protractor, construct a line perpendicular to this segment through P. This will be the tangent to the circle at P. The tangent to a circle at P will always be at a 90 degree angle to a radius of the circle that ends at P.

You can also use a compass to quickly construct a tangent line to a circle. Check it out here: http://www.mathopenref.com/consttangent.html. Instead of just drawing little arcs, make a full circle with its center at Q. Notice that angle RPS is an angle in a semicircle. As we saw earlier, such an angle is always 90 degrees, which is why this method gives us a perfect tangent line.

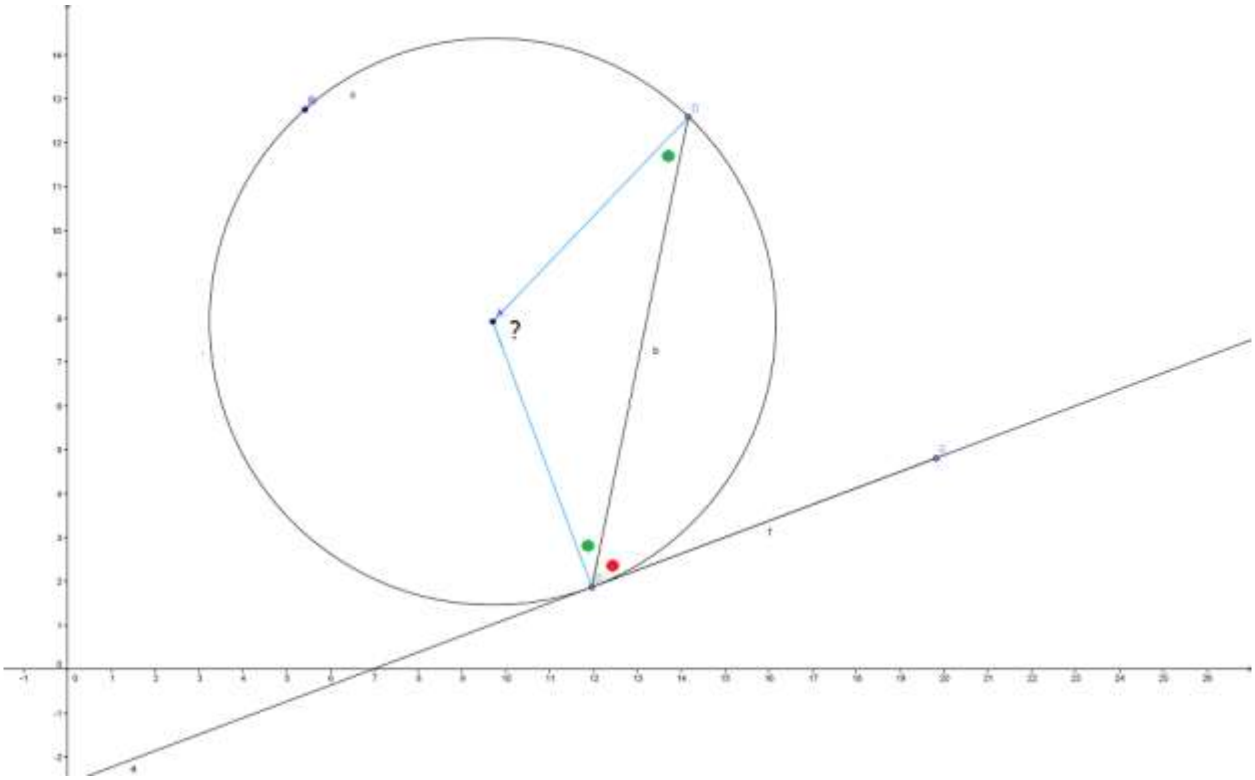
Now that we know how to draw a tangent, we will explore the angle created by a chord and a tangent. As you might expect by now, such an angle has a special property.

Draw a circle in Geogebra. Create a point C on the edge of the circle. From the menu of the perpendicular line button (4th from the left), select Tangents and follow the instructions to create a tangent at point C. Now place a random point D on the circle and draw a chord from C to D. Next, create point E on the tangent line to the right of D. Create the line segment DE, which will be placed on top of the tangent line. This segment will allow us to measure the angle between the tangent and the chord. Click on the angle button and select first the segment and then the chord to mark the angle. We are interested in the measure of the intercepted arc. Create some colored line segments so you can measure the central angle associated with this arc. Record your two angle measurements. Move point D around and take 3 more measurements. Place point D so that CD is a diameter of the circle and take one more measurement.



Analysis

1. The angle between the chord and the tangent is acute:



The angles with the green dots are equal because they are inside a/an ____________ triangle.

a. right
b. obtuse
c. isosceles
d. equilateral

What is the sum of the angle with the green dot and the angle with the red dot?

_____ degrees

What is the sum of the two angles with the green dots and the angle with the question mark?

______ degrees

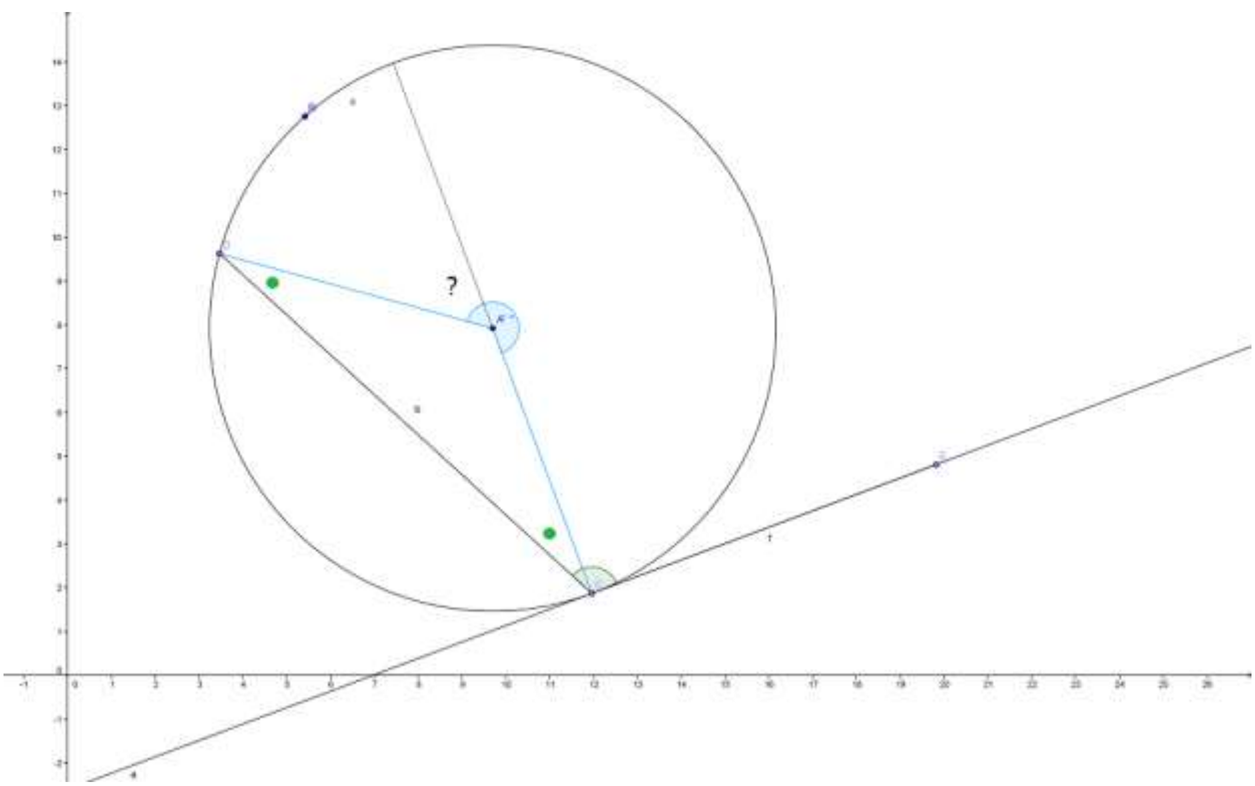
What combination of colored dots belongs in the angle with the question mark?

a. two green dots
b. two red dots
c. a red dot and a green dot
d. two red dots and two green dots

This tells you that the intercepted arc is _____ times the angle between the chord and the tangent

.

2. The angle between the chord and the tangent is obtuse.



What combination of colored dots belongs in the angle with the question mark?

a. two green dots
b. two red dots
c. a red dot and a green dot
d. two red dots and two green dots

The measure of the angle between the chord and the tangent is indicated by the small green arc at point C. The measure of this angle is the green dot + ____ degrees.

The angle of the intercepted arc is indicated by the small blue arc at the center of the circle. The measure of the intercepted arc is 180 degrees plus the dots that you chose to put in place of the question mark. This means that the measure of the intercepted arc is ____ times the angle between the chord and the tangent.

Intersecting Chords

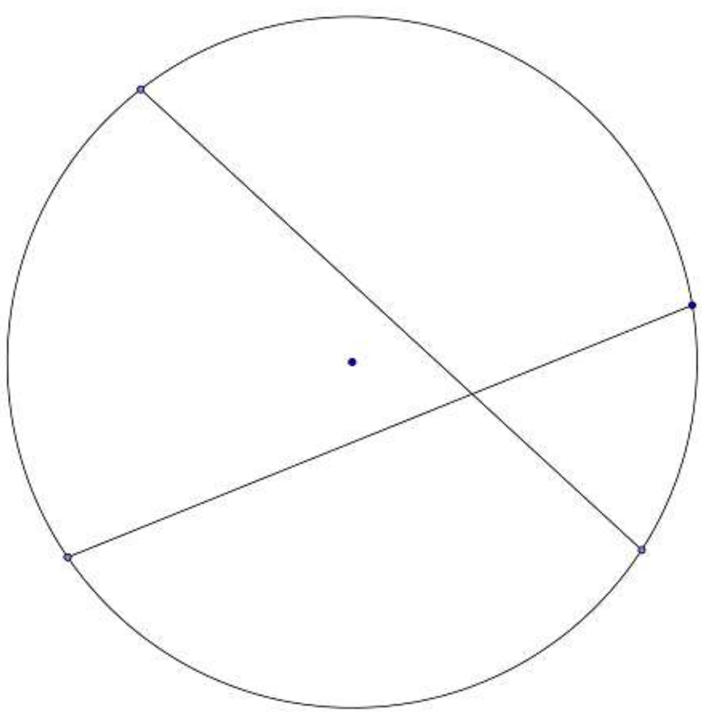
A chord is a line segment with its endpoints on the edge of a circle. When two chords intersect, something interesting happens.

Materials

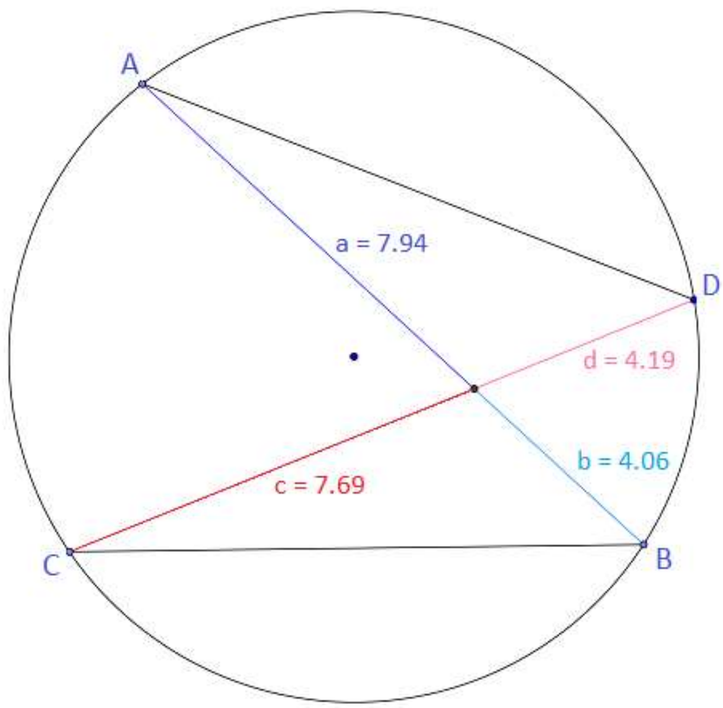
Geogebra

Procedure

First draw this picture in Geogebra:



Then use the menu of the second button to mark an intersect point where the two chords cross. We will not need the actual chords, so right click on them and select "show object". This will make them disappear. Next, use the menu of the third button to create separate segments where the chords used to be, like this (point and segment labels have been changed for illustration purposes):

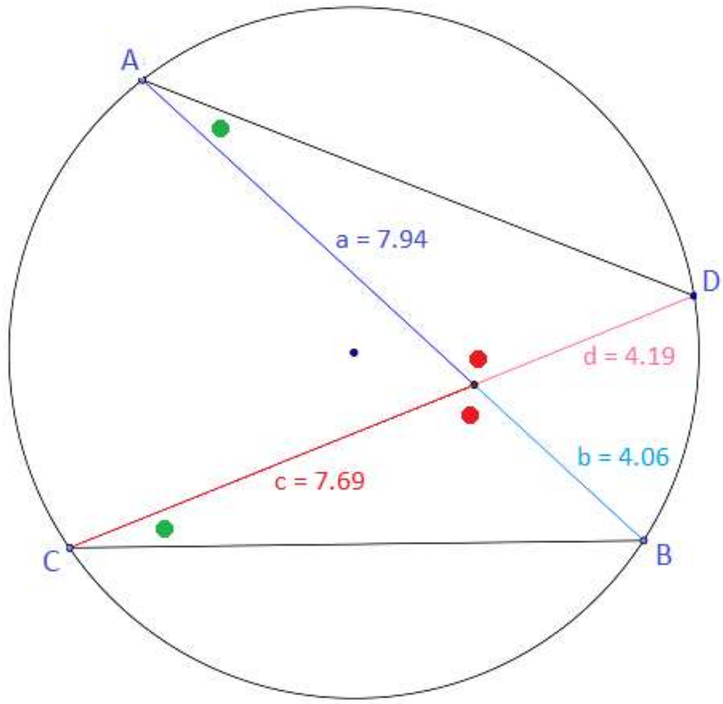


You can get the nice colors by right-clicking on the segments and selecting "object properties". Go to the colors tab to pick out a suitable color. Mark the length of each segment using the menu of button 8.

Multiply the lengths of the two blue segments (7.94 times 4.06 in the picture) and also the lengths of the red and pink segments (7.69 times 4.19).

Move your points around and multiply the segment lengths again. Take at least three measurements. Create a table that shows your results. What do you notice?

Analysis



Look back at the previous experiment to see why the angles labeled with the green dots are equal. The angles with the red dots are vertical angles (see the experiment "Vertical Angles"). When we know two angles of a triangle, we really know all three because they have to add up to 180 degrees. Two triangles that have the same angles are similar. Similar parts of similar triangles are in the same proportion to each other (see the experiment "Angles of a Triangle".)

In the picture above, segment a (the dark blue segment) corresponds to segment c (the red segment), and d (the pink segment) corresponds to b (the light blue segment). Because the proportions must be the same, we can say that $\frac{a}{c} = \frac{d}{b}$. We can cross-multiply two equal fractions to get a times b equals c times d. If you haven't seen cross-multiplication before, just try it out with any two equal fractions like $\frac{1}{2} = \frac{2}{4}$.

ab = cd

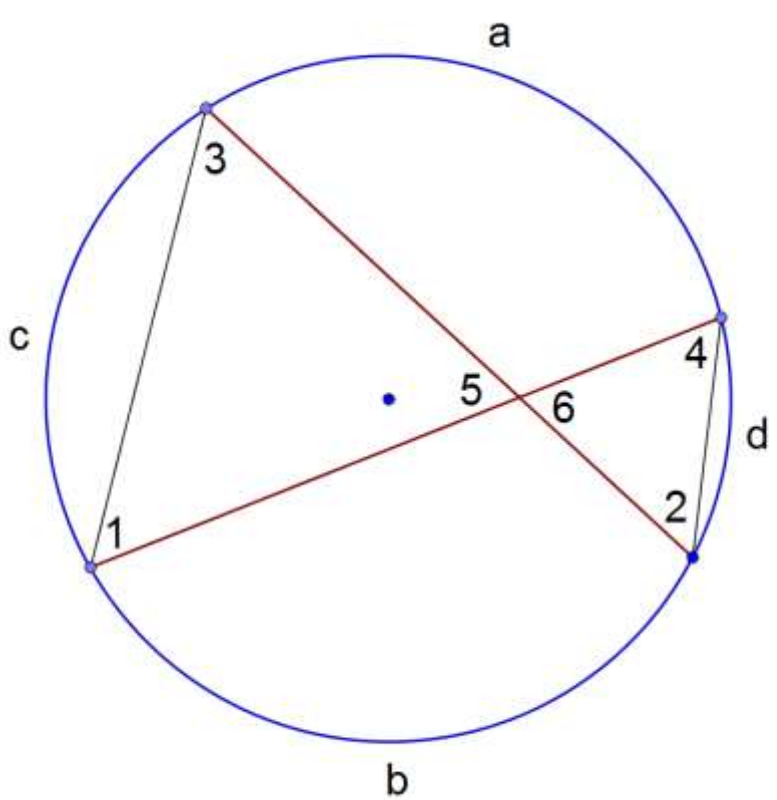
Two Angles in a Circle

Materials

Protractor
Ruler
Paper
compass
Geogebra

Procedure

Using your compass and a ruler, copy the picture shown below. Create a similar picture in Geogebra.



The sum of angles 1, 3 and 5 is ___________

The sum of angles 2, 4, and 6 is ___________

If m∠ indicates the measure of an angle, then

m∠1 + m∠2 + m∠3 + m∠4 + m∠5 + m∠6 = ____________

The 360 degrees of the circle are made up of the individual arcs a, b, c and d. So we can say that the measures of the arcs sum to 360 degrees: $a + b + c + d = 360^o$

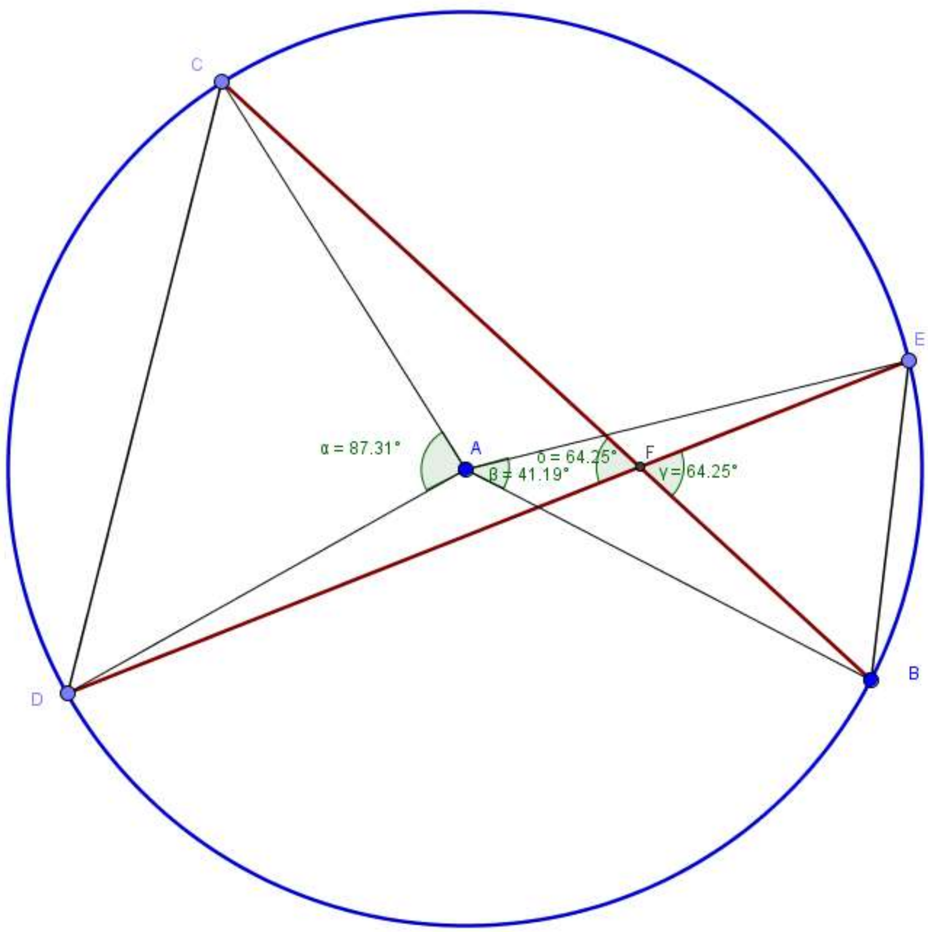
Using Geogebra, verify that angle 1 is congruent to angle 2. Why is that? Explain it in terms of the relationship of these angles to arc a. How is the size of angle 1 related to arc a?

Angles 3 and 4 have a similar relationship to arc b. Verify that they are also congruent.

Angles 5 and 6 are congruent because they are ________________ angles. Check the sizes of these angles in Geogebra.

Geometry textbooks claim that the sum of angles 5 and 6 is equal to the sum of the two corresponding arcs, c, and d. Of course you shouldn't believe stuff just because it's printed in a textbook. Do you think that is true?

Let's check it out in Geogebra. Use the angle measurement button to determine the size of angles 5 and 6. We can't measure the arcs intercepted by these angles directly, but we can measure the central angles. The picture below shows where I have drawn some black line segments to measure these central angles. The degree measure of an arc is the same as the degree measure of the central angle. My arcs measure 87.31^o and 41.19^o, while angles 5 and 6 are both 64.25 degrees. Do your own measurements to see if the sum of the angles created by the intersecting chords is always the same as the sum of the measures of the intercepted arcs.



Analysis

In slightly simplified notation it looks like this:

$\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6 = 360^o$

$a + b + c + d = 360^o$

Things that are equal to the same thing are also equal to each other:

$\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6 = a + b + c + d$

The measure of angle 1 is equal to half the measure of the intercepted arc a, and so is the measure of angle 2:

$\angle 1 + \angle 2 = a$

The measure of angle 3 is equal to half the measure of the intercepted arc b, and so is the measure of angle 4:

$\angle 3 + \angle 4 = b$

By the substitution property of equality we can say that

$a + b + \angle 5 + \angle 6 = a + b + c + d$

Because of the subtraction property of equality it is true that

$\angle 5 + \angle 6 = c + d$

The sum of the two angles inside the circle is equal to the sum of the measures of the intercepted arcs.

An Angle Outside of a Circle

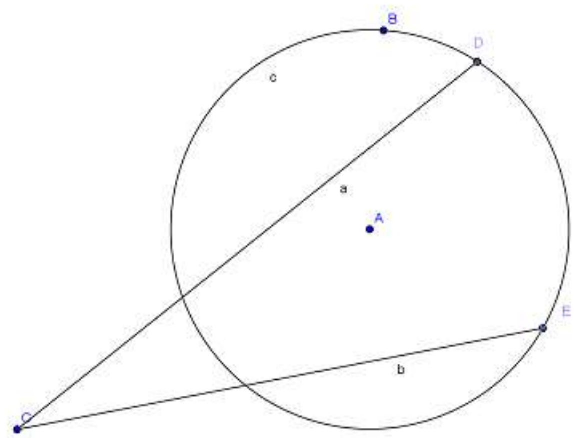
We've seen some interesting () facts about angles inside circles. But what if the angle is outside the circle, is there still something we can predict?

Materials

Protractor
Ruler
Paper
Geogebra
compass

Procedure

Draw the following picture in Geogebra:

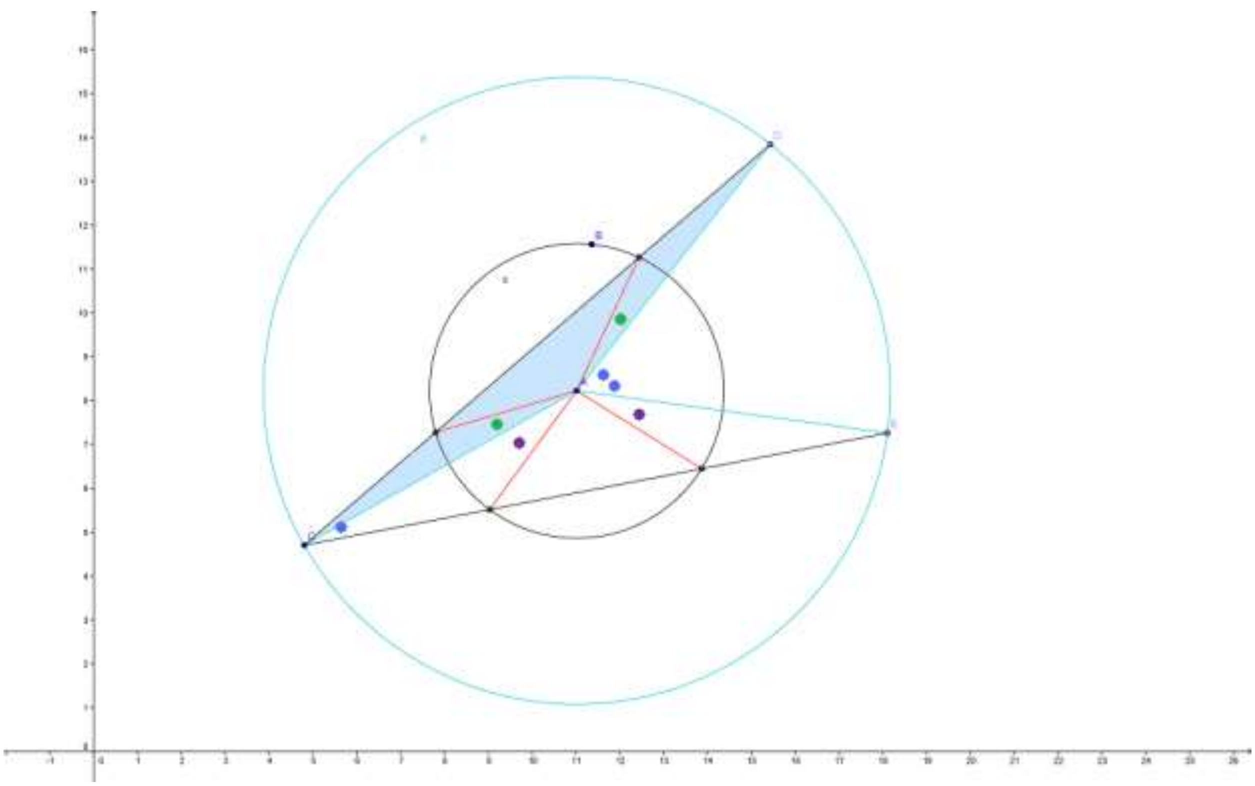


This angle has its vertex outside of the circle. It intercepts two arcs of the circle, a smaller one and a larger one. There must be some kind of relationship between the size of the angle and the size of the intercepted arcs, as you can see when you play around with your Geogebra picture.

As we saw before, there is no way to measure the arcs directly, so you need to draw some colored lines just like you did in the last experiment. Take at least four different measurements of the angle and the intercepted arcs, and record them in a table.

Analysis

It would be difficult to find the relationship by looking at your results. However, we can discover it by adding a few things to our picture. Notice that I have added a new circle, with the same center, that passes through the vertex of the angle outside the original circle. This gives us an important advantage, because the angle is now an inscribed angle in the blue circle. I have marked some blue dots to show what this tells us.

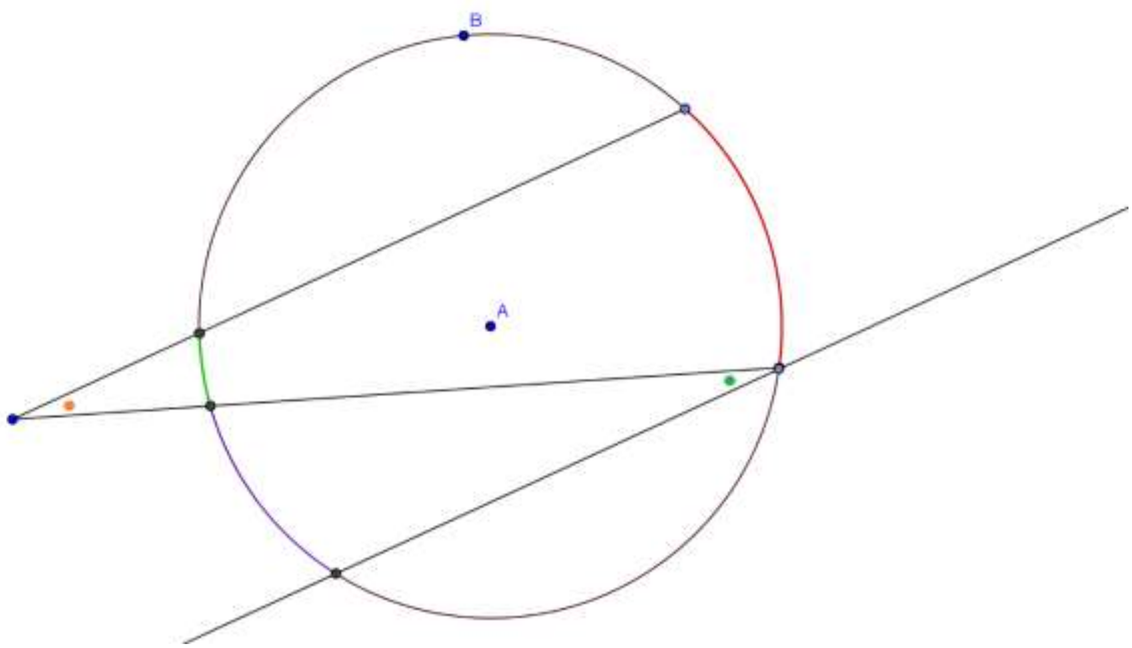


Our new picture has some nice isosceles triangles. The larger triangle that is shaded blue is isosceles, so its base angles are equal. Inside the blue shaded triangle is a smaller isosceles triangle with red sides, and its base angles are equal too. The angles marked with the green dots are equal because the triangles that contain them are congruent. These triangles each have a red side and a blue side, which are radii of a circle. They each have one of the base angles of the shaded triangle, and an angle that is supplementary to the base angle of the isosceles triangle with the red sides. The angles with the purple dots are equal for the same reasons. The angle with the blue dot is inscribed in the larger circle, so we can mark its central angle with two blue dots. Look at the difference between the two intercepted arcs of the smaller circle: it is two blue dots.

Now we can state the relationship: The measure of the outside angle is one-half the difference of the measures of the intercepted arcs. Check your data and see if this relationship is indicated by your measurements.

Play around with the intercept points, and see if you can place them so that the relationship does **not** hold.

It is also possible to prove this relationship by drawing a line parallel to one of the sides that makes up the angle:

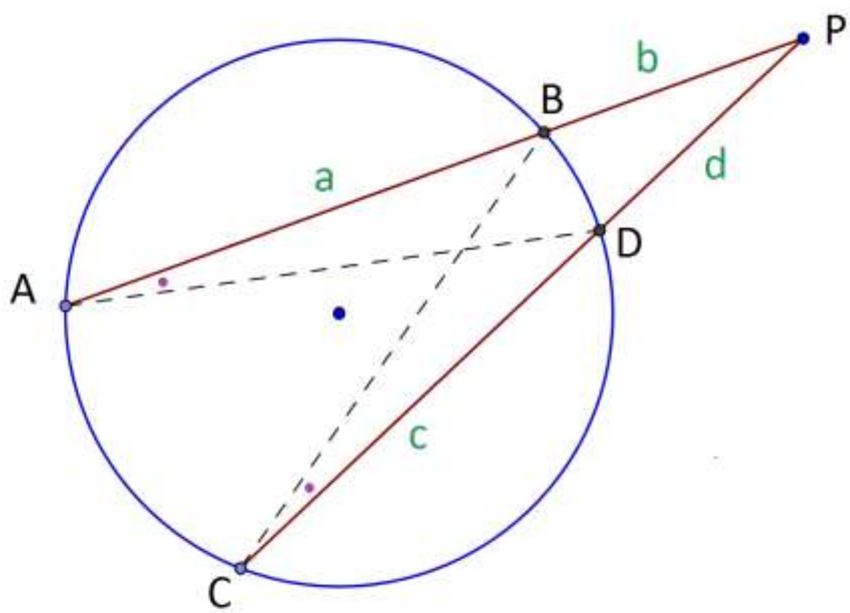


You can prove that the angle marked with the orange dot is equal to the angle marked with the green dot, and that the degree measure of the purple arc is two green dots. Write a proof showing that the measure of the angle marked with the orange dot is half the difference of the degree measures of the red and the green arcs.

More Circle Theorems

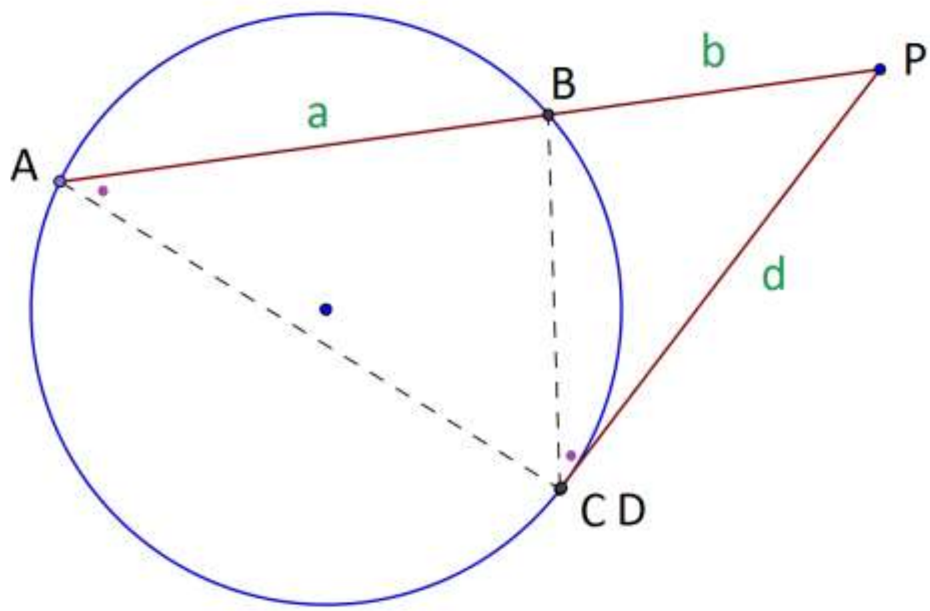
In the image below, two line segments with a common origin P intersect a circle. The following relationship holds: The part of the first segment that lies outside the circle, multiplied by the entire length of the segment, is equal to the part of the second segment that lies outside the circle multiplied by its entire length:

$b(a + b) = d(c + d)$



To see the principle behind this, look at triangles ADP and CBP. The angles at A and C (marked with purple dots) are equal because they intercept the same arc. Also, the triangles share the angle at P, which means that all of their angles are the same. Triangle ADP is not actually congruent to triangle CBP, but these two triangles are similar. We can create some similar ratios using corresponding parts of these triangles. a + b in triangle ADP corresponds to c + d in triangle CBP. d in triangle ADP corresponds to b in triangle CBP. The ratio $\frac{a+b}{c+d}$ must be the same as the ratio $\frac{d}{b}$: $\frac{a+b}{c+d} = \frac{d}{b}$. Cross-multiply to get $b(a + b) = d(c + d)$

In this next picture, I have moved the lines around a bit. Line segment CP has become a tangent to the circle, and point C is sitting on top of point D.



Not surprisingly, the same relationship still holds. The length of segment c is now 0:

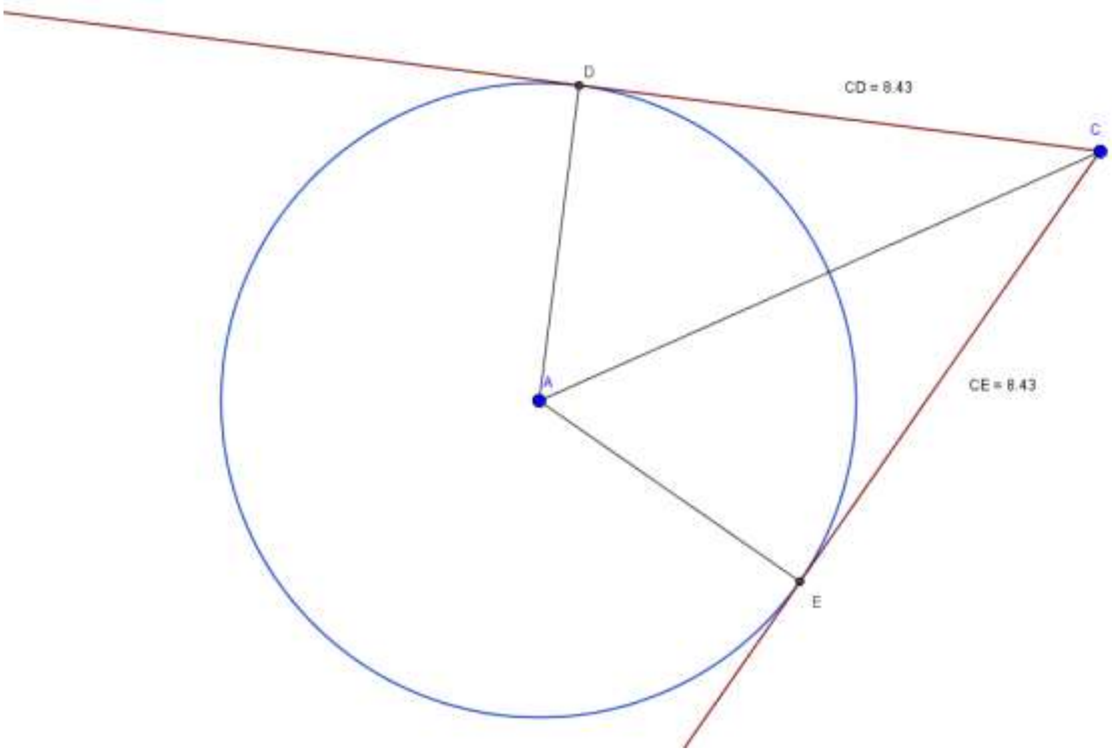
$b(a + b) = d(0 + d)$, or $b(a + b) = d^2$

The proof however is slightly different. We need the conclusion from the experiment "A Chord and a Tangent", which was that the angle between the chord and the tangent is one-half the intercepted arc. So, the angle between chord CB and tangent CP, marked with a purple dot, is ½ the measure of arc CB. That makes it the same size as the angle at A. Triangle ADP (the big triangle) and triangle CBP (the smallest triangle) also share the angle at P, so they are still similar as they were in the previous case. The longest side (a + b) in the large triangle corresponds to d, the longest side in the small triangle. Side d in the large triangle is opposite the angle at A, and corresponds to b in the small triangle. The ratios are equal:

$\frac{a+b}{d} = \frac{d}{b}$. Cross-multiply to get $b(a + b) = d^2$.

If we also place point A on top of point B, both segments are tangent to the circle. Length a would be zero, and the formula now says that b(0 + b) = d(0 + d). That means that both segments will be equal in length, since b^2 is equal to d^2.

In this picture, tangents to the circle have been drawn from point C. You can see that segment CD is congruent to segment CE.



Write a proof that shows that CD is congruent to CE. Use information from the experiment "How to Balance a Line on a Circle" to determine the size of angle ADC and angle AEC.

Understanding Volume

Volume works much like area, except that we measure volume in 3-dimensional units rather than in square units. 3-D units are cubic units, like cm^3 or in^3. One cubic inch ($1\ in^3$) is a cube with sides of 1 inch. When we measure the volume of some 3-dimensional shape, we are trying to find how many little cubes fit inside it. If the shape is a box, that is fairly easy to do. Consider a small box that is 8 inches by 10 inches on the bottom, and 5 inches tall. If we cover the bottom with little 1 inch cubes, we can place 8 rows of 10 cubes, so 80 cubes fit on the bottom. Because the box is 5 inches tall, 5 layers of 80 cubes can fit in the box. The total volume is 5 x 80 = 400 cubic inches. This tells us that we can find the volume of a box by multiplying the width times the length times the height. The formula for that is

V = WLH

If we know the volume of a box, but not the width, we can rearrange the formula quickly by realizing that LH, or the length times the height, is just a single number once you multiply the two quantities: V = W x LH. Again we can use the simple example 6 = 2 x 3, and pretend that we don't know the 2. We can get the 2 by dividing both sides of the equation by 3: $2 = \frac{6}{3}$. In the same way, $W = \frac{V}{LH}$.

The volume of a box can be also be determined by first finding the area of the base (length times width), and then multiplying by the height. This works well for boxes that don't have a rectangular base. For example, if a box had a triangular base, you could find the area of the triangle and then multiply it by the height of the box. You could do the same thing if the box had a base shaped like a hexagon. "Boxes" with variously shaped bases are called prisms in geometry. Sometimes the prism is shown lying on its side, but you can still find the volume just as you would for any box.

The volume of a prism equals area of the base times the height: V = BH

Materials

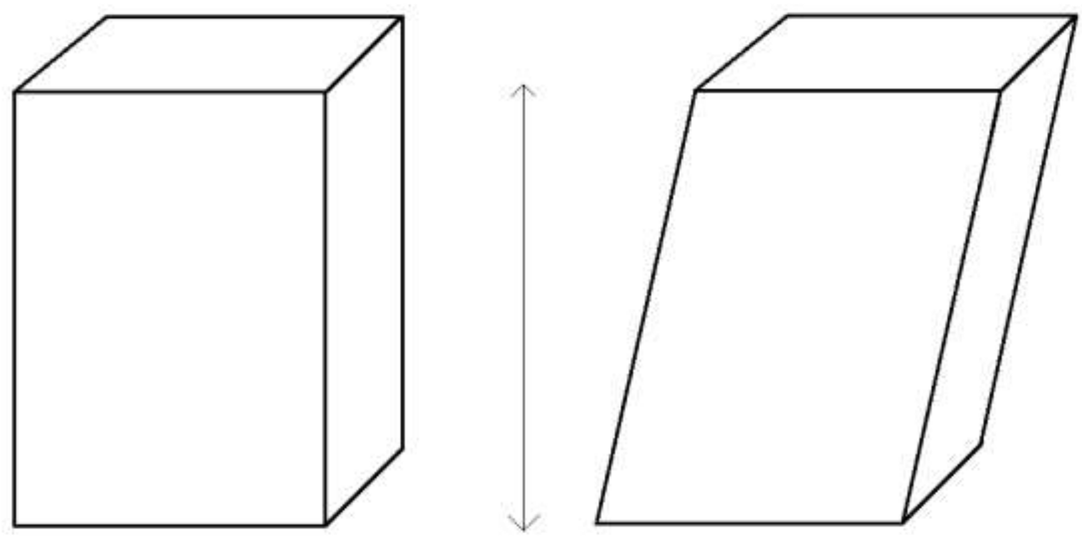
Ruler with cm markings

Tall flat box, like a cereal box
Marker
Stack of square or rectangular crackers
Tape (optional)

Procedure

1. Determine the volume of your box. Use the centimeter markings on your ruler, and measure to the nearest millimeter (the small markings). If the box is too tall for your ruler, carefully make a mark at 30 centimeters, and then measure the remaining height. Record the volume. What unit is used to indicate the volume?

Some prisms are slanted. The two boxes shown below have the same height, length and width.



Use a stack of crackers to create a shape that looks like a rectangular box. Slide each cracker over a bit so that the stack looks slanted, like the box in the picture. Does this change the height of the stack? Does it change the amount of food? Do you think the volume of the two boxes shown above is the same?

Determine the surface area of your box. Record it and show your calculations. What unit is used to indicate the area?

Next, we will divide our box in half along its longest length. Mark the midpoint of all four longest sides and draw lines connecting those midpoints. Now imagine cutting the box in half along these lines, and placing both halves on top of each other to form a new box. It is not necessary to actually cut the box, but if you want to do so try not to stab scissors into the lines because you could hurt yourself. Start your cut from the top end of the box toward the line, and then tape things back together as needed. The new box would be more compact than your original tall and thin box. The sides would be more equal in length than they were before. Would the volume still be the same? Determine the surface area of the new box. Record it and show your calculations.

Compare the surface area and volume for each box. Divide the surface area by the volume. Which box has a larger surface area to volume ratio?

Next, we will imagine shrinking our original box. Take each measured side and make the value 10 times smaller (divide each measurement by 10). Calculate and record the volume and surface area of the imaginary small box. Which box has a larger surface area to volume ratio: the larger box or the smaller box?

Analysis

Actually, we can repeat this experiment in two dimensions to make it easier to see what is going on. Compare a rectangle with sides of 5 cm and 20 cm with a square that has sides of 10 cm. Both shapes have the same area, but which one has the larger perimeter? Next, compare a rectangle with sides of 0.5 cm and 2cm to your original rectangle with sides 5 cm and 20 cm. Which rectangle has a longer perimeter compared to its area? (Divide the perimeter by the area).

You can shrink a box by using algebra. Call the width W, the length L, and the height H. The volume is WLH. Now make each value 10 times smaller. The new volume is 0.1W times 0.1L times 0.1H, which is equal to .001WLH. The volume of the shrunken box is 1000 times smaller than that of the original box. The original surface area is 2WL + 2WH + 2HL. Using factoring, you can write that as 2(WL + WH + HL). The new surface area is 2(0.1W)(0.1L) + 2(0.1W)(0.1H) + 2(0.1H)(0.1L) = 0.02WL + 0.02WH + 0.02HL or 0.02(WL + WH + HL). The surface area of the shrunken box is 100 times smaller than that of the original box. This means that the shrunken box has a relatively larger surface area. The surface area to volume ratio is 10 times that of the original box.

Surface area to volume ratios are important in many areas, especially in biology because it affects the size of cells.

Such ratios are also important in marketing. Goods are often packaged to maximize the surface area to volume ratio to make consumers think that they are getting more for their money.

The Surface Area and Volume of a Cylinder

1. Surface Area of a Cylinder

A cylinder is a shape that looks like a soup can or a soda can. It can be straight or slanted. In terms of volume, a cylinder is just a box with a round base. Just as you can cut a box apart and then find the surface area of the pieces, you can cut and flatten a cylinder to determine the surface area. Because that is a bit hard to do with a metal can, we'll use a smarter method.

Materials

An ordinary can, like a soup can
Paper (prefer construction paper)
Scissors
Permanent marker
Ruler
Tape

Procedure

For this exercise, we will pretend that our soup can is a perfectly smooth cylinder, and calculate its surface area. First trace the bottom and the top of the can on a sheet of paper. Next, make a small mark along the top edge of the can. Lay the can on its side on a sheet of paper, with the mark touching the paper. Mark the paper at this point. Then carefully roll the can along the paper until the mark on its edge again touches the paper. Mark the paper here and draw a line between the two points you marked on the paper. If you are using a large can you may need to tape two sheets of paper together. Measure the height of the can. Draw two lines on the paper with the same length as the height of the can, at the endpoints of the first line and perpendicular to them. Complete the rectangle and cut it out. Verify that it fits perfectly around the can just like a label.

Now you have 3 shapes that together represent the surface area of the can. Arrange them so that they touch each other, creating a shape that could be folded up to create a 3-D paper copy of the can. A two dimensional shape that can be folded into a three dimensional object is called a *net*. While some people find it easy to make such nets and see immediately which 3-D shapes they represent, I sometimes have trouble imagining things in 3-D. It is also potentially confusing that different nets can be folded into the same 3-D shape. Put some tape on your net to hold it together, and make sure that you can fold it into a cylinder.

Next, take one of the circles, or make a new one, and carefully fold it in half. Use the crease to measure the diameter of the circle. This is probably the most accurate way to find the diameter of a relatively small circle. Measure the length of the diameter. Now you can calculate the surface area. Do not forget to consider both the top and the bottom of the can.

Here is a nice animation: http://www.mathopenref.com/cylinderarea.html

Analysis

Rolling a can on paper and creating a rectangle to represent the side, or *lateral*, surface area seems like lot of trouble. How would you calculate the area by measuring only the diameter of the can and its height?

Checkpoint

Go to http://www.phschool.com/webcodes10/index.cfm?fuseaction=home.gotoWebCode&wcprefix=auk&wcsuffix=0099 and take the lesson quiz for Chapter 1, Lesson 1.2.

2. Volume of a Cylinder

When you calculate the volume of a box, you multiply the length times the width times the height. That is the same as taking the area of the bottom and multiplying it by the height. In the same way, you can calculate the volume of a cylinder by taking the area of the bottom and multiplying it by the height: http://www.wikihow.com/Calculate-the-Volume-of-a-Cylinder

Go to http://www.mathopenref.com/cylindervolume.html, and adjust the image to create a cylinder with a base that has radius 10, and a height of 15. Calculate the volume and compare it with the volume listed on the website. Show your calculations. Now click “Allow oblique” so you can create a slanted cylinder. Use the same radius and height. Is the volume still the same?

Create a cylinder using a stack of identical coins, like pennies. Slide each of the coins over just a little so your stack becomes a slanted cylinder. Does it look like the volume should still be the same? Why or why not?

The Surface Area and Volume of a Cone

Materials

Protractor
Ruler with centimeter markings
Construction Paper
Compass
Tape
Uncooked Rice

Procedure - Surface Area

Draw a large circle on construction paper. Carefully measure the radius of this circle (to the nearest millimeter), and calculate its area (see Understanding Area). Record your measurement and calculations. If you made the circle by tracing a round object instead of using a compass, you can measure the diameter, which is the largest distance across the circle. Then divide by 2 to get the radius. Next, draw a central angle of 90 degrees in the circle. Cut out the circle very carefully, and then cut away the central angle so that you have 3/4 of a circle left.

Fold the circle into a cone, and tape the edges together. Do not overlap the edges. There are only two measurements that you can make easily when you have a cone. It is not so easy to measure exactly how tall the cone is, so instead we get the slant height s by putting a ruler along the side of the cone. Again try to measure very accurately, to the nearest millimeter. Notice that s is really the radius of the original circle. Also measure the diameter of the bottom of the cone (put a ruler across the bottom, and record the largest measurement). The radius of the cone is one-half the diameter. Record your findings.

What is the surface area of your cone?

Analysis - Surface Area

The surface area of this cone is rather easy to find, because we already know that it was made out of exactly 3/4 of a circle. Its surface area is 3/4 of the area of the original circle. The radius of the circle was s, the slant height of the cone, so the area is 3/4 times πs^2. If you have not already done this calculation, do it now and record the result. Most of the time we can't find the surface area of a cone this way because we don't know what portion of a circle was used to make the cone. We have to figure it out indirectly, using only the slant height of the cone and its radius.

Look at your measurement of the radius of your cone, and calculate its circumference, which is $2\pi r$. The circumference of the circle you drew originally was $2\pi s$, where s is the slant height of the cone. You should find that the circumference of the cone is 3/4 of the circumference of the original circle.

No matter what section of a circle we remove to create a cone, we can always look at the ratio between the two circumferences to find the surface area of the cone: (circumference of cone/circumference of circle) times (area of circle) = area of cone.

When we take $2\pi r$ and divide it by $2\pi s$, 2π cancels out and we are left with r/s. Try that out with your own measurements: r/s should be very close to 3/4 (0.75).

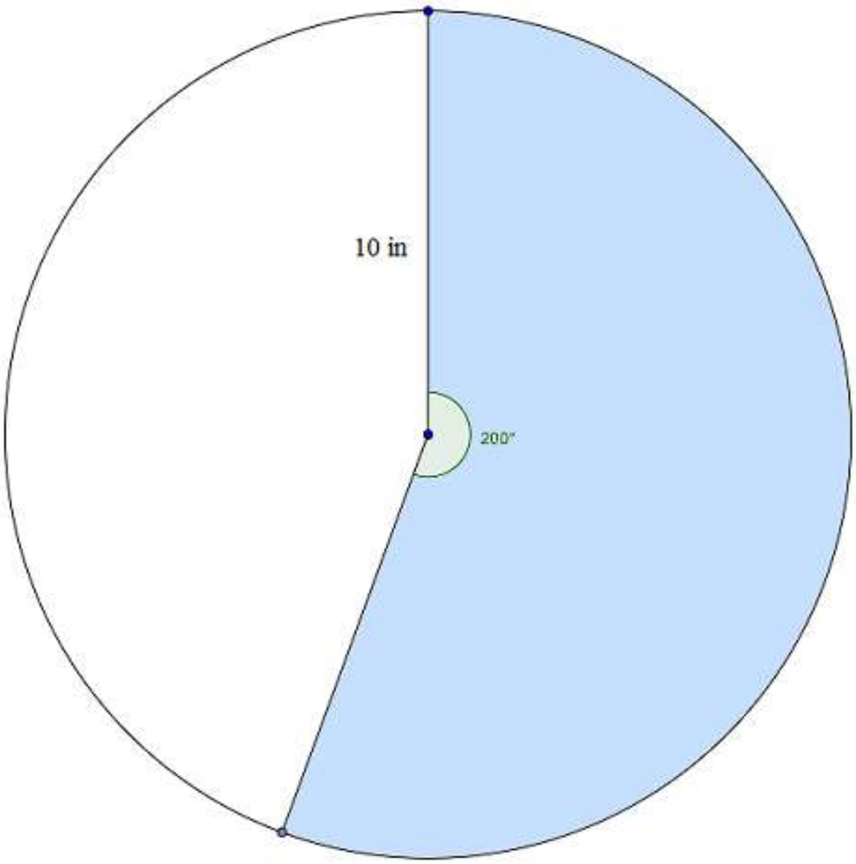
So that means that if we have the slant height of a cone, and its radius, we can find its surface area using the formula: r/s times the area of the original circle, which is $\frac{r}{s}$ times πs^2.

Because multiplying by s and dividing by s cancels out, the actual formula becomes: πrs. Use this formula with your measurements. Do you get the same area you got earlier? It may not be exact but it should be fairly close.

Be careful - our cone has no bottom, while the ones you encounter on tests usually do! You must remember to add the surface area of the bottom: Surface area of cone = $\pi rs + \pi r^2$. The part of the surface without the bottom is called the *lateral surface area*.

If the radius and slant height of a cone are given you can just use the formula. Go to the quiz to see what to do when a problem doesn't provide all of this information.

Question 1



What is the lateral surface area of the cone that can be made from the shaded part of the circle? Leave your answer in terms of π.

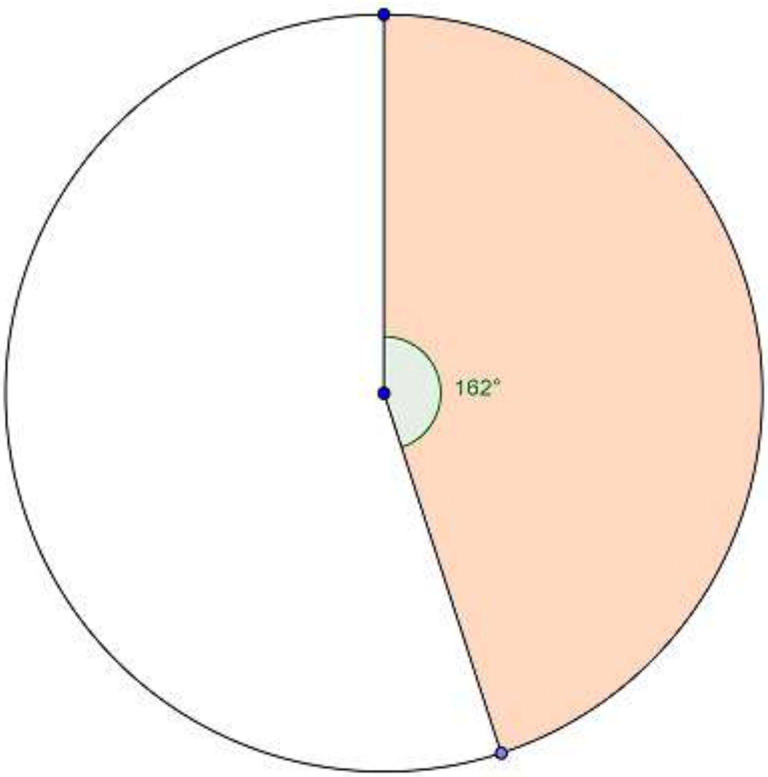
First we need to find the area of the whole circle. We use the formula for the area of a circle, πr^2, to get ☐ π in^2.

Next we want to know what portion of the circle is used to make the cone. The whole circle has ☐ degrees, and the shaded portion is 200 degrees. The part of the circle used for the cone is ☐ / ☐ which is approximately 0.56.

The surface area of the cone will be ☐ π in^2 (round to the nearest whole number).

Now try one by yourself: **The shaded area in the figure above is increased so that it occupies 216 degrees. The surface area of the cone that can be made from this shaded area is now** ☐ π in^2.

Question 2



The cone that can be constructed with the shaded part (162 degrees) of the circle has a surface area of 22.5 in^2. Find the slant height and radius of the cone.

First we need the total area of the circle. The shaded area divided by the total area is ☐ degrees/☐ degrees = ☐. This means that the area of the circle is ☐ in^2.

If we know the area of a circle we can find the radius. Use the formula for the area of a circle and round to the nearest whole number. The radius of the circle is [ ] inches.

The radius of the circle becomes the [ ] of the cone.

To find the radius of the cone we need the circumference of the cone. The circumference of the cone is [ ] times the circumference of the circle (enter a decimal number). The circumference of the circle is [ ] inches (round to the nearest whole number). The circumference of the cone is [ ] inches (round to the nearest whole number). That means that the radius of the cone is [ ] inches (round to 1 place after the decimal point).

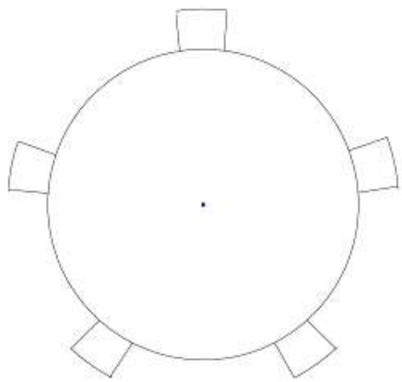
Now check your answers using the formula for the surface area of a cone. The calculated surface area is [ ] (round to 1 place after the decimal point). This should be very close to 22.5 in^2, the area given in the problem.

Procedure - Volume

Next, we will determine the volume of a cone. For this, we want a cone that is a little more cup-shaped. On construction paper, draw a circle with a 4 inch radius (8 inch diameter). Draw a diameter and cut out half of the circle. Overlap the edges about 1 inch to create a cone. Tape the cone together so it is sturdy. Carefully measure the diameter of your cone.

Next, we want to create a cylinder that is exactly as tall as the cone. Because it is difficult to measure how tall the cone is, use the Pythagorean Theorem to calculate the height (see experiment 13). A cross-section of a cone can be drawn as two right triangles, just like the cross-section of a pyramid. Note that the slant height, which should still be 4 inches if you measured right, is the "C" side of your triangle, while the radius - half of the diameter- is the "B" side.

On construction paper, draw a circle with this same diameter. Next, draw another circle with the same center that is about a cm larger. We want to cut out the inner circle, but leave about 5 small tabs.



This will be the bottom of your cylinder. Take another sheet of construction paper to make the rest of it.   If you did your calculations right, your cylinder should be exactly as tall as your cone - put them side by side to see if it looks like they are the same height.

Next, fill your cone with rice, exactly level to the top. Use the dull side of a knife and draw it across the top of the cone to make sure the rice is level with the edges. Carefully transfer the rice in the cone into the cylinder. Notice that the cylinder is not full. Fill the cone with rice again, and put this rice into the cylinder too. Is the cylinder full now? How many times can you put a cone-full of rice into the cylinder?   This is known to be a whole number, but slight errors in your constructions can cause the cylinder to be a bit too small or too large to fill exactly.

Analysis - Volume

The volume of a cone is ... of the volume of the cylinder with the same base and height.

This is somewhat difficult to prove, but we will look at the equivalent proof for the volume of pyramids in the next experiment. Euclid uses this proof to explain the volume of a cone.

The Volume of a Pyramid

To understand what is going on in this experiment, you need to know what a prism is. If you have not done so already, read this explanation from Sparknotes: http://www.sparknotes.com/math/geometry1/geometricsurfaces/section2.rhtml

Because the concepts in this experiment involve three-dimensional figures, it is sometimes difficult to imagine that the volume of one shape equals the volume of a somewhat different shape. However, researchers have determined that our brains are hard-wired to be able to accurately estimate the amount of food in front of us, no matter how it is arranged. We will therefore use food to help us understand how the volumes of different shapes can be equal.

Materials

Clean Protractor
Clean ruler
Sharp knife
4 Slices of bread
20 Square crackers

Procedure

Create a stack of square crackers, about 10 crackers tall. This is a right prism. Next, create a second stack of 10 crackers where each cracker is moved about 1/16 inch to the left. This also forms a prism (if you imagine that the sides are perfectly smooth). Note that both stacks are equally tall, even though the second stack is slanted. Both stacks have the same base and top. Now, imagine that you are very hungry but can only choose one of these stacks of crackers to eat. Your brain will instantly tell you that both shapes contain the same amount of food. Someone might say that we are cheating here, because a real prism isn't made of 10 crackers. However, if we had a huge amount of time we could build these same shapes out of squares of paper, or imagine that they were made of even thinner slices than that. This approach is used in calculus, and it works very well to determine volumes. The volumes of both prisms are in fact the same.

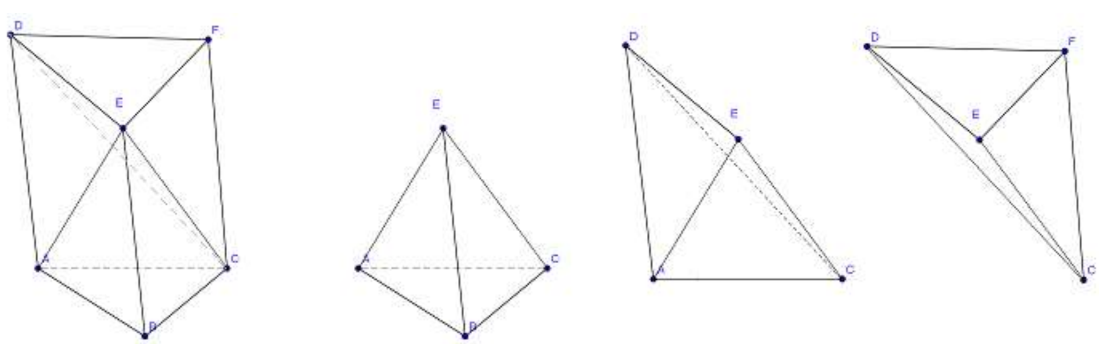
Now we will do the same thing for pyramids. Our pyramids will not be smooth, but the results are the same as if we used "infinitely thin" slices.

Cut a square with sides of 7 cm from a slice of bread. Cut another square with sides of 6 cm, and so on, until you get to a 1 cm square. Stack all the squares neatly on top of each other so that the smallest square sits on top in the middle. This is a regular pyramid, kind of like the ones in Egypt. Now take your pyramid apart and put the slices so that one corner of them lines up exactly, which leaves the smallest piece sitting way off to one side of the pyramid instead of in the center. People don't usually build lopsided pyramids like this, but they are the key to understanding the volume of a pyramid. We can line our slices up along any corner and the volume is the same as if the pyramid was symmetrical.

Next, cut all of your squares diagonally, so that each square is divided into two identical triangles. Take one set of these triangles and build a pyramid that has the longest edges lined up, with the smallest triangle in the center along that edge. (This would look as if you had cut your first symmetrical square pyramid in half.) Build another pyramid by lining up the corners at the 90-degree angle. Notice how these two pyramids are different, but they are made of the same amount of bread so they have the same volume. In fact, regardless of where the top is located, any triangular pyramid that has the same base and the same height has the same volume.

Analysis

Look this picture, which shows a triangular prism before and after it is cut into three triangular pyramids:



It is probably easiest to see that the first pyramid, with base ABC and top E, has the same volume as the third pyramid, which is upside down with base DEF and top C. Triangle ABC is equal to triangle DEF because these triangles form the top and bottom of the prism. If you carefully consider where these two pyramids are sitting inside the prism, you can see that they must both have the same height.

Now look at the last two pyramids from a different perspective, as if their bases are at the back. The second pyramid then has base ACD, and the third has base DCF. However, these triangles together form the back wall of the original prism. Because that back wall is either a rectangle or a parallelogram (slanted rectangle), the cut line DC divides it exactly in half so the two bases have to be equal. Next look at the tops, which is now the point E for both pyramids. Because of where E is located relative to the base of both pyramids, they have the same height. Therefore their volumes are equal. If the volume of pyramid 2 is equal to that of pyramid 3, and pyramid 3 has the same volume as pyramid 1, then all three must have equal volumes.

This is the proof that Euclid uses to show that the volume of a triangular pyramid is 1/3 that of the equivalent prism. A rectangular pyramid is simply two triangular pyramids put together, and a rectangular prism is two triangular prisms put together. The volume of a rectangular pyramid also equals 1/3 that of the corresponding rectangular prism. You can carry this further for pyramids and prisms with any polygon as their base, because these polygons can be divided up into triangles.

Show that the volume of a pyramid with a height of 10” and a square base with sides of 5” is $83\frac{1}{3}$ cubic inches.

A triangular pyramid is 20 cm tall. Its base is an equilateral triangle with sides of 6 cm. Show that the volume of this pyramid is approximately 104 cubic centimeters.

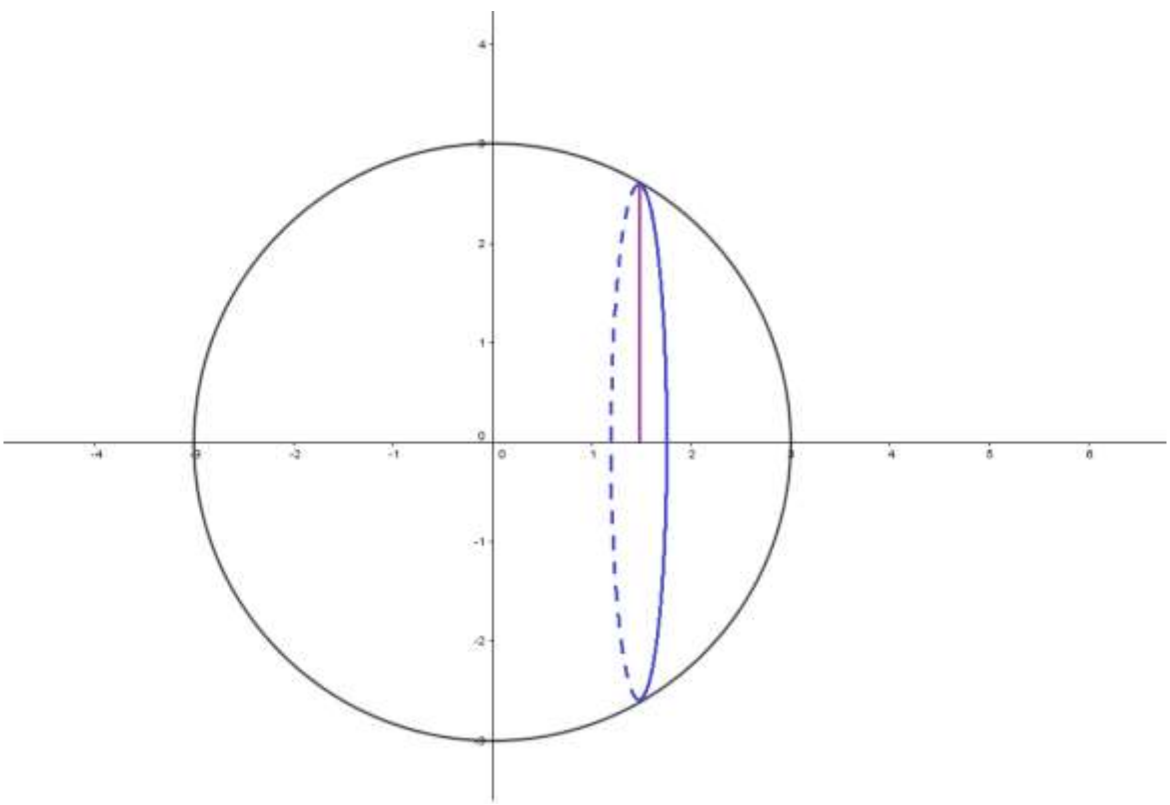
If you believe that a pyramid with a 16-sided regular polygon has a volume of 1/3 that of the equivalent prism, it is not too far-fetched to think that a cone-shaped pyramid would have a volume that is 1/3 that of the equivalent cylinder. Euclid has an actual proof of this:
http://aleph0.clarku.edu/~djoyce/java/elements/bookXII/propXII10.html

This site has a nice animation showing the volume of different pyramids:
http://nrich.maths.org/public/viewer.php?obj_id=1408

The Volume and Surface Area of a Sphere

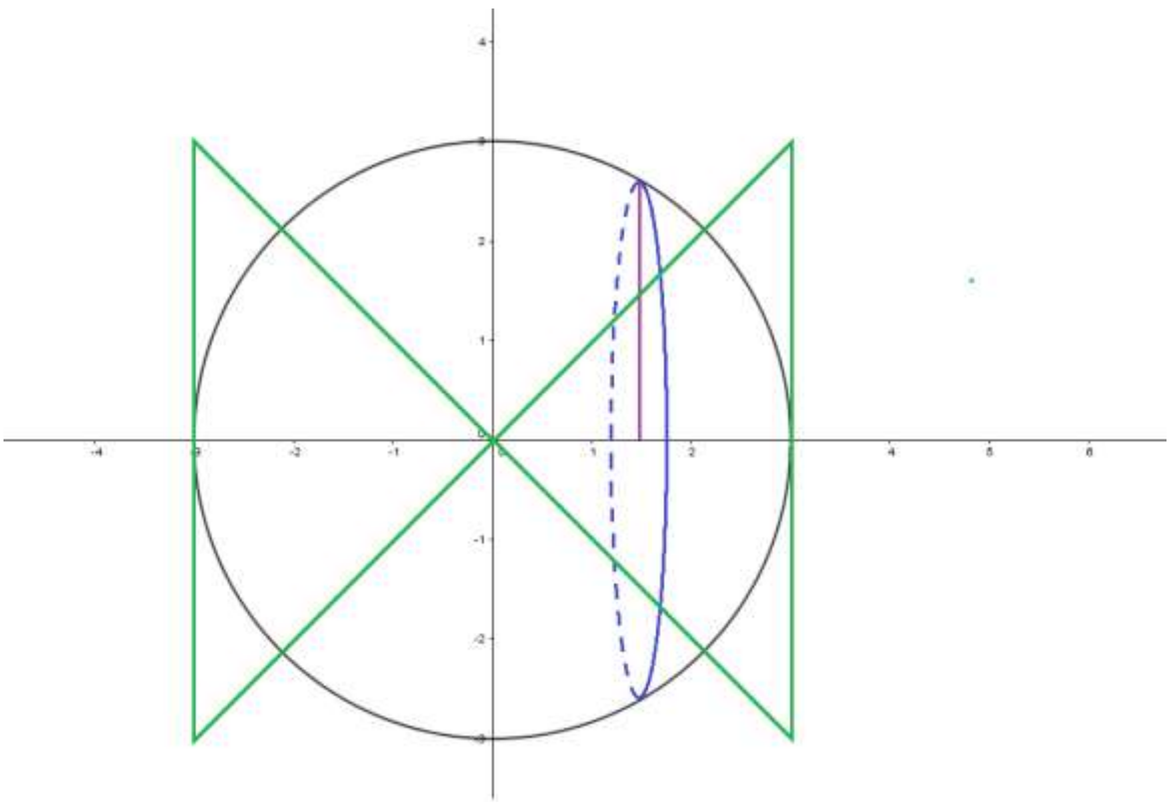
The formula for the volume of a sphere was discovered by Archimedes in ancient Greece over two thousand years ago. Archimedes accomplished this amazing feat by using a method very similar to modern calculus.

Imagine taking a sphere, and cutting it up into vertical slices. Each slice would be round, but the edge would be just a little bit slanted. Now imagine that you could make each slice incredibly thin – as thin as possible. Such slices are called "indivisibles", because they considered to be so thin that they cannot be divided further. At this point the slant on the edge is no longer significant. All we need is the radius of each slice, and we could calculate the volume.



If you look back at "Going in Circles", you will see that the equation of a circle is $x^2 + y^2 = r^2$. This describes the outer edge of the sphere, projected onto a flat surface. Here r is the radius of the sphere, but the radius of the slice is y. $y^2 = r^2 - x^2$. We could take the square root to find y, but we just need the surface area of the circular slice which is π times the radius squared (y^2).

This means that each slice of the sphere has an area of $\pi(r^2 - x^2)$, which is $\pi r^2 - \pi x^2$. Even though r, the radius of the sphere is just a number, that still looks really complicated. Using $r = 3$ like in the picture above, we get that each slice has a surface area of $9\pi - \pi x^2$. Of course that is different for each slice. The bigger x gets, the smaller that cross-sectional area is, and each slice will have a different volume. So now what?? You and I would probably just give up at this stage, but Archimedes kept working. If only he could add something so each slice would be the same. That something turned out to be a cone. Although Archimedes' formal prood shows a single cone, we will use two smaller ones.

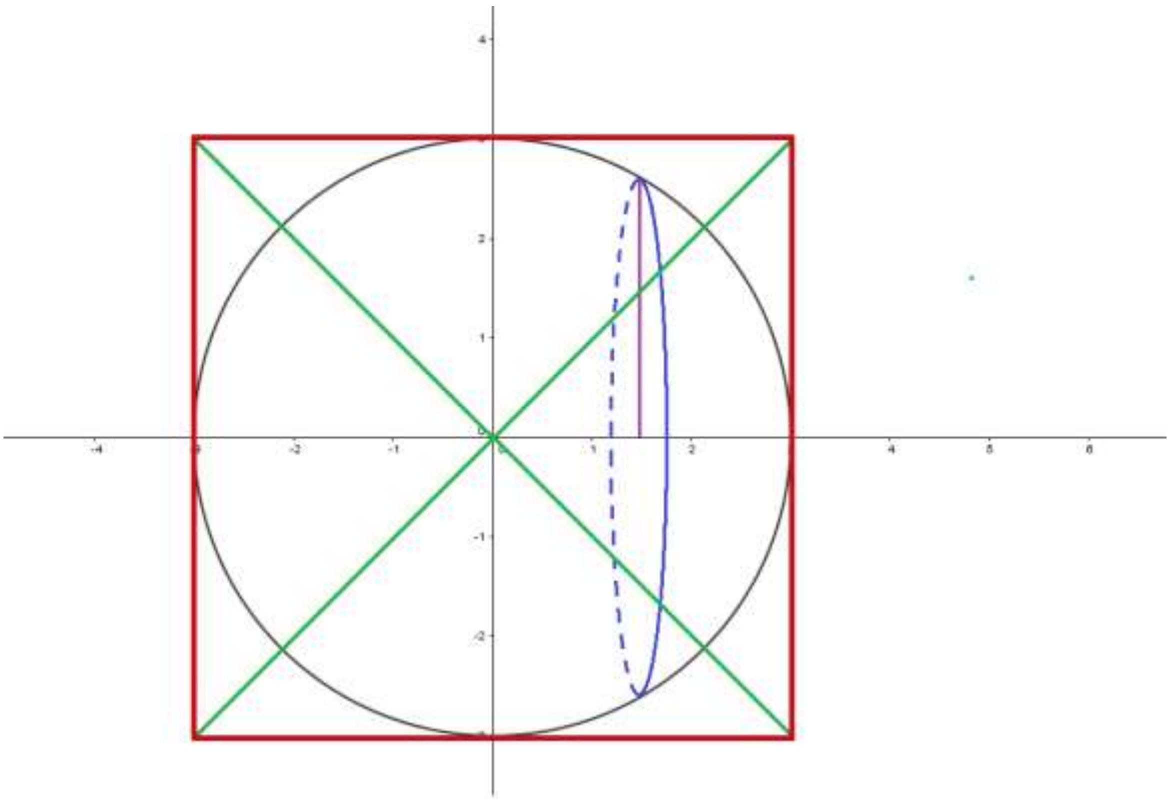


This image shows a flat projection of the two cones, lying sideways. Each cone has a radius of 3 units, just like the sphere, and a height of 3 units. We can slice these cones vertically just like we did with the sphere. The radius of each cone is determined by the equations of the slanted green lines, $y = x$ and $y = -x$. Each slice of the cone will have a different volume, and the radius of each slice is y. That means the cross-sectional area is πy^2, which is the same as πx^2.

For each location x, we can add the cross-sectional areas of the slice of the sphere and the slice of the cone:

$9\pi - \pi x^2 + \pi x^2 = 9\pi$

Notice that this doesn't depend on x, so it doesn't matter where you cut your slices. The combination of the two slices will always have the same total area, and if they have the same thickness the volume will be the same! Now it is easy to add them up to get the total volume. It is just like taking round slices with a radius of 3, and putting them all together to stretch across from x = -3 to x = 3. Hmm, that would be a cylinder, with a radius of 3 and a height of 6:



This image shows the flat projection of the cylinder. This is the smallest cylinder that completely encloses the sphere. Its volume is $\pi r^2 h$, which in this case is 9π times 6 or 54π.

The two cones plus the sphere have the same volume as the cylinder! Because we know how to find the volume of cones and cylinders, we can calculate the volume of the sphere shown in this example. Show that the volume of this sphere, with a radius of 3 units, has to be 36π.

Archimedes used his discovery to find a general formula for the volume of a sphere, and we can do the same.

The volume of a cylinder with radius r and height 2r is : ___________________

A cone with radius r and height r has a volume of : ___________________

The volume of the two cones is: ___________________

This means that the volume of a sphere is ___________ minus ____________, or

____

Use this formula to show that a sphere with a radius of 3" has a volume of 36π in^3, which is approximately 113 in^3.

Note that 2 cones of radius r and height r have the same volume as one larger cone with radius r and height 2r. If you take a cone, a sphere, and a cylinder that have the same height and radius, the volume of the cone plus the sphere equals the volume of the cylinder.

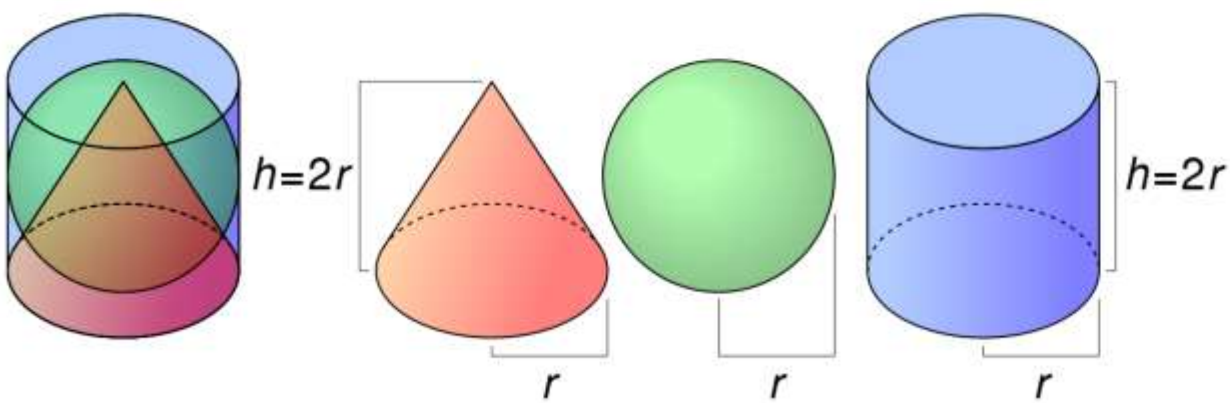


Image: Cmglee, Wikimedia Commons

After Archimedes found the volume of a sphere, he wanted to know its surface area. Instead of dividing the sphere up into really thin slices, he now imagined that it was filled with thousands of little cones, each with its base on the surface of the sphere, and its tip at the center. The height of all of these little cones would be r, the radius of the sphere. If you take that apart, you can put the little cones upright on a table and push them all together into a circle. Those cones would be a tiny bit wobbly, because their base would be just a little bit curved. But if you made those cones incredibly small so their bases would be like pixels on your computer screen, that wobble wouldn't be significant, and you'd simply have a circle with lots of thin tall spikes on it.

Earlier, we learned that slanted pyramids and cones have the same volume as straight ones, so long as you keep the height the same. You'll have to stretch your imagination a bit, but you can slant all those small cones toward the middle of the circle until you have one large cone with a height of r. The volume of this large cone is still the same as the volume of the sphere, and **its base is the same as the surface area of the sphere.**

Because the cone has a height of r, and a volume equal to that of a sphere with radius r, you can find the base.

First, let's try that out with some real numbers. The volume of a sphere with a radius of 5 cm is $\frac{500}{3}\pi$ cm^3, or approximately 524 cm^3. Now consider a cone with the same volume that is 5 cm tall. The volume of the cone is 1/3 · base · height (1/3 of the volume of the corresponding cylinder), so the base must have an area of approximately 314 cm^2, which is also the area of the sphere's surface. Show your calculations to confirm this.

Next, do the same thing with a sphere of radius r, and find the formula for its surface area. Use your formula to show that a sphere with a radius of 5cm should have a surface area of about 314 cm^2.

Archimedes then realized that the surface area of a sphere is the same as the lateral surface area of a hollow cylinder that fits exactly around the sphere (the cylinder has no top or bottom). The diameter of the cylinder would be the same as that of the sphere, and it would be exactly as tall. Show your calculations that confirm this.

Here is a video that demonstrates the size of the surface area of a sphere by peeling an orange:

http://www.youtube.com/watch?v=VvFYZLpMbR4

Today we can easily confirm Archimedes' findings by using calculus, but what he did so long ago was truly an amazing achievement.

The Distance between Parallel Lines

For those times when you just need to know the distance between two parallel lines... Actually this type of problem is not commonly found on tests. However, it does illustrate some important principles that you should know, which is why it is included here.

To understand this material, you need to be familiar with the standard coordinate system, the slope of a line, and the equation of a line. See "Finding Your Center: The Orthocenter" for a review.

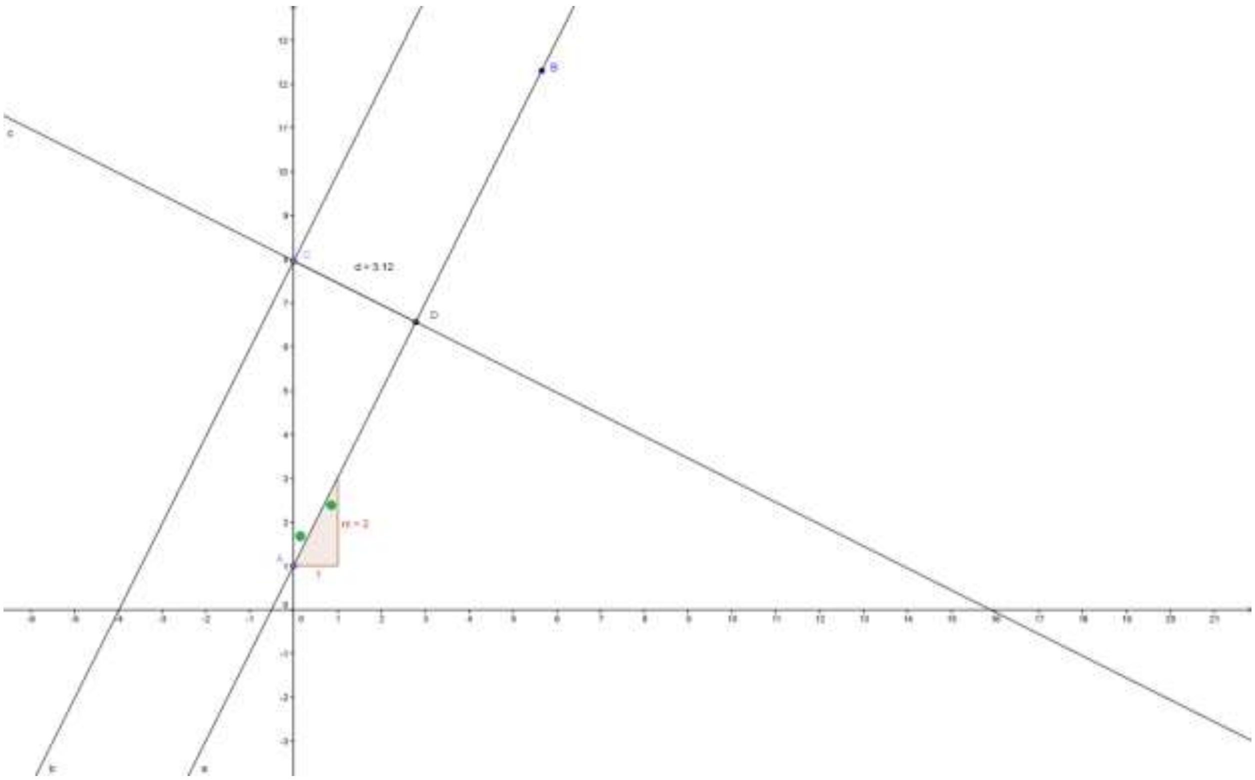
Materials

Protractor
Ruler
Graph Paper (paper with little squares on it) or Geogebra

Procedure

For this experiment you will need to work with a coordinate system. If you have some graph paper, draw the x-axis and the y-axis and label them. Choose a convenient value for the slope m of your first line, and a nice value for b_1. I chose 2 for the slope and 1 for the first intercept. Then draw a second line parallel to the first (so it has the same slope) with a different y-intercept point. Notice that you cannot easily measure the distance between these two lines with a ruler. It either takes several tries to find the shortest distance, or you need to make sure that you are measuring at a 90 degree angle to both lines.

If you are drawing in Geogebra, you need to place a y-intercept point, and one additional point to get the first line. Then place the point that you want for the second y-intercept. Use the menu on the 4th button from the left to create a line parallel to the first line through this point. To measure the distance between the lines, you can create a perpendicular line at the highest intercept point, as I have done in the picture below. Mark the intersect point D using the menu on 2nd button from the left. The only way to measure the distance is by actually creating a line segment here, even though there is already a line there. I have created a line segment between C and D and measured it using the menu on the angle button.



Analysis

Mathematically, there are two entirely different ways to calculate this distance.

First, we can make use of the fact that when you multiply the slope of two perpendicular lines, you always get -1. Don't take my word for it, check it out yourself in Geogebra. So, if my slope m is 2, then I already know that the line through C and D has slope -1/2. I also know the y-intercept of this line, which is at point C. Therefore b is 8. This gives me the equation of the line: y = -1/2 x + 8. The line through A and D has equation: y = 2 x + 1. If you know algebra, you can find the intersect point D from these two equations.

Once you know the coordinates of point D, you can find the distance between C and D using the Distance Formula, which is really just the Pythagorean Theorem.

However, that's algebra, and we are doing geometry here. Geometrically, we can solve this problem rather easily. As an observant geometry student, you have probably seen that there are two triangles in the picture, so you immediately start to compare their angles. There are in fact two angles there that are alternate interior angles. The "Z" is reversed and lying on its side at the little triangle on the bottom, and the angles marked with green dots are equal. This means that the two triangles have at least one angle in common. But wait, they are both right triangles, so they have two angles in common. What does that tell you about the third angle? Hmm, looks like these two triangles have the exact same angles. This means that they have similar proportions. If two triangles are similar like that, you can obtain the bigger triangle from the smaller one by just multiplying each side by the same number. Therefore, you can divide one equivalent side by another and you always get that same number. (See “Are Triangles With the Same Angles the Same?)

Note that the side marked d=3.12 in the big triangle corresponds to the side marked 1 in the small triangle. The slanted side of the big triangle is the distance between the intercept points. This corresponds to the slanted side of the small triangle, which by the Pythagorean theorem is $\sqrt{m^2 + 1^2}$, and that is the same as $\sqrt{m^2 + 1}$.

Using my sample values of 8 and 1 for the two y values of the intercept points, we can set this up as a proportion:

$$\frac{d}{1} = \frac{8-1}{\sqrt{m^2+1}}$$

Grab a calculator and check it out for m = 2. Geogebra displays this value correctly to the nearest millimeter (the last digit, representing tenths of a millimeter, is not accurate enough). Repeat your calculation using the values in your own drawing, and see how close you get.

Transformation Target Practice

Old-style computer games often had a simple drawing representing a person or creature that you could move around the screen. The same figure would appear to go up and down or sideways, sometimes flipping around to face either left or right, as seen in below in screenshots of “Dangerous Dave”.

For two-dimensional games, the figure is always the same size. To simulate three dimensions, we want our figures to appear smaller or larger to indicate how far away they are. All of these changes can be accomplished using transformations. Moving Dangerous Dave around a computer screen is fun, but in geometry we use much simpler shapes like triangles and rectangles that are easy to draw.

Materials

Geogebra

Procedure

1. Translation

If you just need to move a figure from one spot to another, a translation will do the job. You have to specify how many units your figure should move in the horizontal direction, and how far it should move vertically. In Geogebra, this is done by specifying a vector. Draw a vector to indicate the desired movement. Simply place two points, and then select Vector between Two Points from the line menu. Where you place the vector doesn’t matter – only the direction and size of it counts. Vectors are usually specified by how many units up or down and how many units to the side they go. < 5, 2 > means a vector that goes 5 units to the right of its origin, and 2 units up. So, if the start of this vector is at (1, 1), its tip would be at (6, 3). < -3, -1 > is a vector that points to the left of its origin by 3 units, and down 1.

The vector that you draw will specify the movement of an object. Let's give it a try.

First, right-click on the drawing surface to find the option to show the grid, or select Grid from the Options menu. Next, place a relatively small circle. Then, draw another circle *of the same size* close to the first circle. Right-click on this second circle and select Object Properties. Pick a color other than black for this circle. Now, your goal is to specify a transformation that will drop a copy of the first circle onto the second circle. If you are successful the second circle will appear to turn black because a new circle has been drawn on top of it. Count the squares carefully to determine the required change in position of the center of the circle, and create a vector to represent the transformation. Select Translate Object by Vector from the Reflect Object about Line menu. Click on the first circle to select it, and then on the vector. If you don't get it right the first time, just try again. [Sometimes the program malfunctions and won't let you select the circle. If so, just hit the Move button and then click on Translate Object by Vector again.]. Record the vector you used as < __ , __>, and describe how the circle moved.

2. Reflection

Make sure the grid is showing. Use the Polygon button to draw a right triangle with vertices at A = (1, 1), B = (5, 1) and C = (1, 3). Place two points D and E at (7, 1) and (7, 4). Use these points to create a vertical line. Then use Reflect about Line to create a mirror image of the triangle.

Place another two points to create an additional vertical line to the right of the mirror image. Reflect the mirror image about this new line. The new image is a translation of the original triangle. How far would you have to move the original triangle to place it on top of the final image? Check your answer by using Translate Object by Vector. How far apart are your two vertical lines? Move the vertical lines further apart or closer together to see if you can find a relationship between the space between the lines and how far the triangle moves.

3. Rotation

Place a random point on the drawing surface, close to your original right triangle. Select Rotate around Point. Rotate the triangle around the point by 180 degrees. Experiment to see if it is possible to rotate the triangle in such a way that the rotation image ends up on top of the mirror image. You can change the degrees, and move the point of rotation. Why does it not fit exactly on top?

Remove the second vertical line that you used for the reflections. Replace it with a horizontal line through the points (4, -1) and (10, -1). Reflect the mirror image of the triangle (the first reflection) over the horizontal line to create a second reflection. Can you now rotate the original triangle so that it ends up exactly on top of the second reflection image? What is the location of the rotation point, and the angle you used?

4. Dilation

A dilation is a transformation that makes a figure larger or smaller. The figure keeps the same proportions. Dilations are always measured from a particular point. Suppose you want to dilate rectangle ABCD manually, by say a factor of 3, from the origin (0, 0). You would need to measure the distance between the origin and the first corner of the rectangle, A. You then multiply this distance by 3, and measure it out along the line that connects the origin and point A. You place point A', the image of point A, three times as far from the origin as point A. Repeat that with all the other points. The new rectangle, A'B'C'D' will be three times as big as the original rectangle. That is a lot of work, and Geogebra can do these things much faster.

Start a new Geogebra file. Create a rectangle by placing points at (3, 3), (3, 5), (7, 3) and (7,5). Create a second smaller rectangle by placing points at (1, 1), (1, 2), (3, 1) and (3, 2). Change the color of the small rectangle to something different. Use the Dilate from Point button to create an image of the larger rectangle that fits on top of the smaller rectangle. To do so, you must place a point from which to start the dilation, and select a scale factor for the dilation. Experiment until the image appears where you want it. Record the location of the point you used, and the factor.

Analysis

1. Translations move an object without changing its size and shape. There are various ways to describe these translations. One popular way is by specifying what happens to a general point (x, y) as a result of the transformation. If you are using the vector <5, 2>, any point will be moved 5 units to the right and 2 units up. We can write this transformation as (x, y) → (x + 5, y + 2).

2. Reflections produce a mirror image of an object. Because of that, it is not practical to describe such a transformation in a simple (x, y) → … format. However, a second reflection (reflecting the image again), will restore the original orientation if you reflect over the same line, or over a line parallel to that. If you are reflecting an object over two parallel lines, you can predict how far it will move. The distance the object moves is always equal to twice the distance between the two parallel lines. Notice that if you use the same line twice, that distance is zero. In general, if you do a double reflection over two parallel lines that are a distance of d units apart, your object will move a distance of 2d units: (x, y) → (x + 2d, y + 2d).

If you are doing a double reflection over horizontal or vertical lines, you can easily find the distance d. If the lines are slanted, it is always possible to calculate the distance between them (see "The Distance Between Parallel Lines).

3. Rotations around a point preserve the distance between that point and the object. The object moves in a circle around the point, as far as you specify by providing the angle. Due to the 90 degree angle between the x and y axes, we can specify the movement of a general point (x, y) for rotations at 90 degree intervals, as you can easily verify yourself by rotating a sample point like maybe (3, 2).

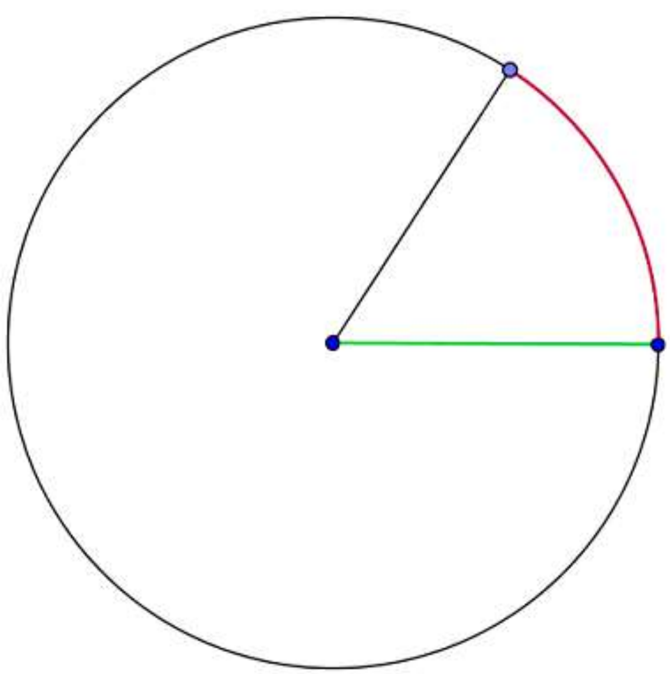
<u>Counter clockwise</u>	<u>Clockwise</u>
90 degrees: (x, y) → (-y, x)	90 degrees: (x, y) → (y, -x)
180 degrees: (x, y) → (-x, -y)	180 degrees: (x, y) → (-x, -y)
270 degrees: (x, y) → (y, -x)	270 degrees: (x, y) → (-y, x)
360 degrees: (x, y) → (x, y)	360 degrees: (x, y) → (x, y)

4. Dilations create a figure that is similar to the original object. Dilations are always specified from a particular point. The distance between that point and the points of the object is changed by the dilation factor, which may be a fraction so that the image is smaller than the original. If you are dilating from the origin by say a factor of 5, it is easy to specify this dilation for a general point: $(x, y) \rightarrow (5x, 5y)$. Both the x-distance and the y-distance increase by a factor of 5, just like the overall distance. If you are dilating from a point other than the origin, you can't just multiply x and y by the dilation factor. The easiest thing to do in that case is to move both your object and the point from which you are dilating, so that the dilation point is at the origin. For example, let's dilate a circle with center (4, 5) and radius 1 by a factor of 3, from the point (-1, -2). First move every point so that the dilation point is at (0, 0): $(x, y) \rightarrow (x + 1, y + 2)$. Now dilate by a factor of 3: $(x + 1, y + 2) \rightarrow (3(x + 1), 3(y + 2))$. This means that each original point (x, y) has changed to $(3x + 3, 2y + 6)$. Then we have to move things back where they were so that the dilation point is now at (-1, -2) again. This means that $(3x + 3, 2y + 6)$ has to move to $(3x + 3 – 1, 2y + 6 – 2)$. The final result is the overall transformation $(x, y) \rightarrow (3x + 2, 2y + 4)$. The radius of the image circle will be 3, since it is three times the size of the original circle.

Trigonometric Ratios

If you play around with circles long enough you will eventually discover the idea of angles. First there is a whole circle. Then there is a half-circle, a quarter- circle, and so on. Angles are a way to divide up a circle into measured parts. Dividing a circle into 360 degrees was a really good idea. The number 360 is naturally very "divisible", giving nice whole number angle measurements for 1/3 of a circle, 1/4, 1/5, 1/6, 1/8, 1/9, 1/10, 1/12 and more. This is a nice simple way to understand angles, and this is how students first learn about them.

The other way to measure out parts of a circle is to relate them to the radius of the circle. This reflects our deeper understanding of the relationship between the radius of a circle, its circumference, and angles. To really understand this yourself, you need to actually work with a circle. Get some paper, a compass, and some thin string. If you do not have a compass, trace a circular object like a plate. The diameter of your circle is the largest distance across the circle. Use a ruler to find that, and then mark the center of the circle in the middle of the diameter. Once you have drawn a large circle, use a string to measure out the radius. Cut your string and lay it around the circumference of the circle. This marks an angle of one **radian**.

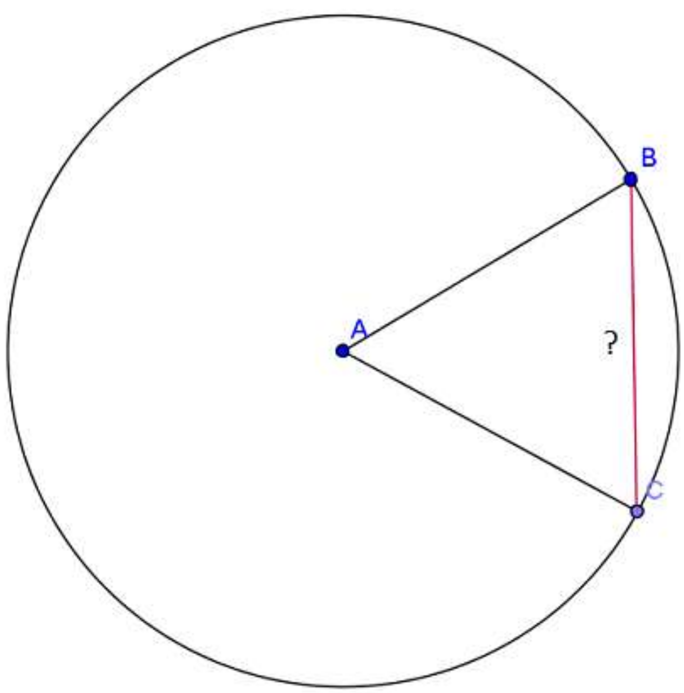


Here one radius is shown in green, and the string along the circumference is colored in red.

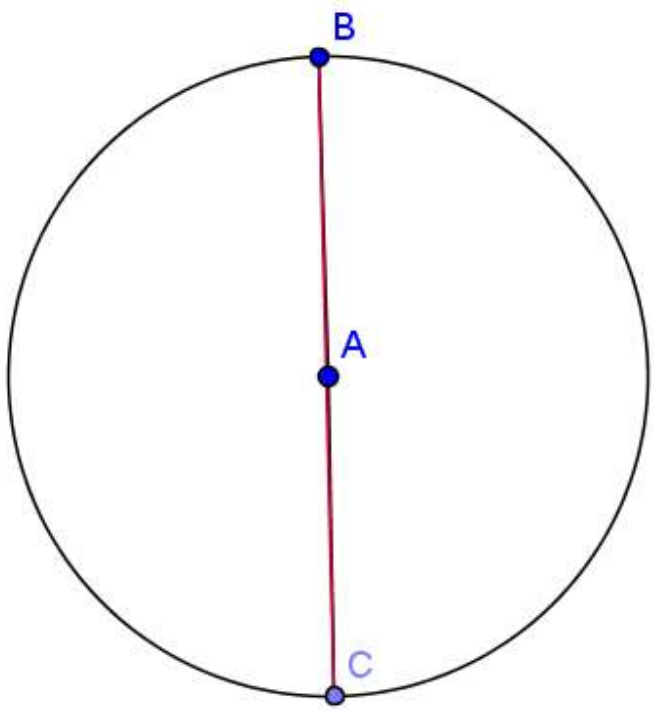
The red string is the same length as the radius. Because you know that the circumference of a circle is $2\pi r$, you can imagine that we can put 2π copies of the red string around the edge of the circle. That is not a whole number. The circumference of the circle is 6 times the radius and a bit more. 6.283185…… radii fit around the circle, so there are that many radians in a circle. We usually just say that there are 2π radians, and 2π radians are equivalent to 360 degrees. That of course means that 180 degrees = π radians. Convert like this: 1 ~~radian~~ times $\frac{180 \text{ degrees}}{\pi \cancel{\text{radians}}}$ is about 57.3 degrees. Look at the picture of a radian again: one radian is a little smaller than a 60 degree angle.

In trigonometry you can work with either radians or degrees. Just make sure your calculator is set to the right mode!! Always check this if you get an unexpected answer from your calculator. The TI 84 has a mode button; just press it and adjust between degrees and radians. Other calculators may show RAD or DEG at the top of the display.

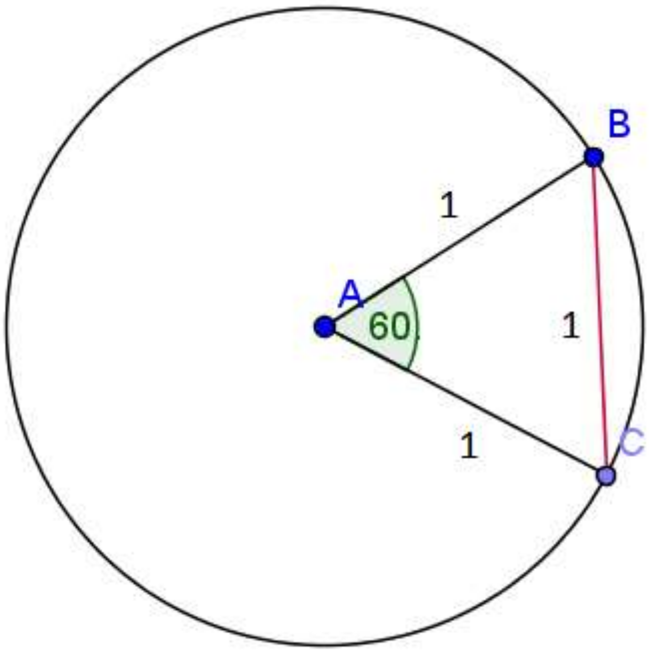
After people had played around with angles for a long time, they started to wonder about chords. A chord is a line segment drawn between two points on the circumference of a circle. Here is a chord associated with a specific angle:



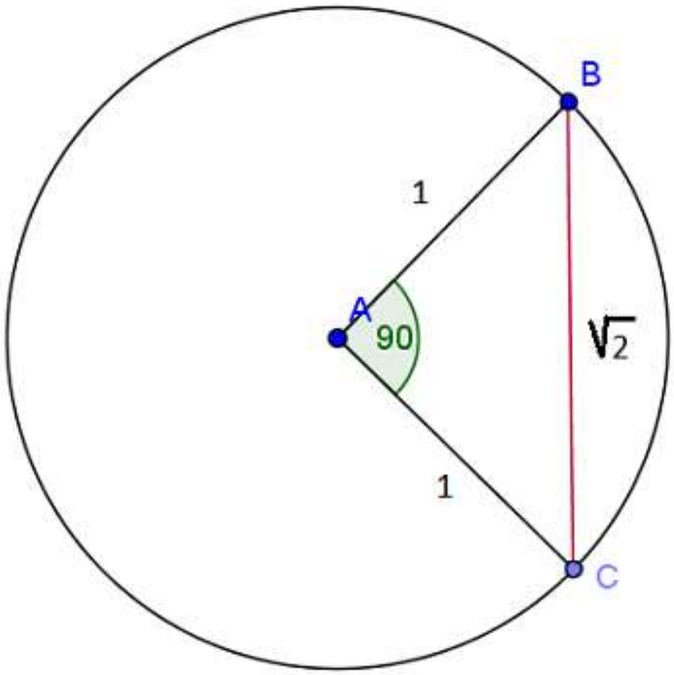
How long is the chord? Well, that depends on the angle, but of course also on the size of the circle. So, if you want to explore the relationship between the angle and the chord you should pick a fixed size for your circle. A convenient size would be to give the circle a radius of 1. "One what?" you might ask. Fortunately that doesn't really matter. It could be 1 inch, 1 centimeter, or even 1 yard. We just say that the circle has a radius of 1 unit. Once we have decided on the size, we can look at different angles and see how long their chords are. The first one is obvious: an angle of 0 degrees has a chord of length 0. The next one is rather simple too: an angle of 180 degrees has an associated chord of 2 times the radius, or 2 units in our case.



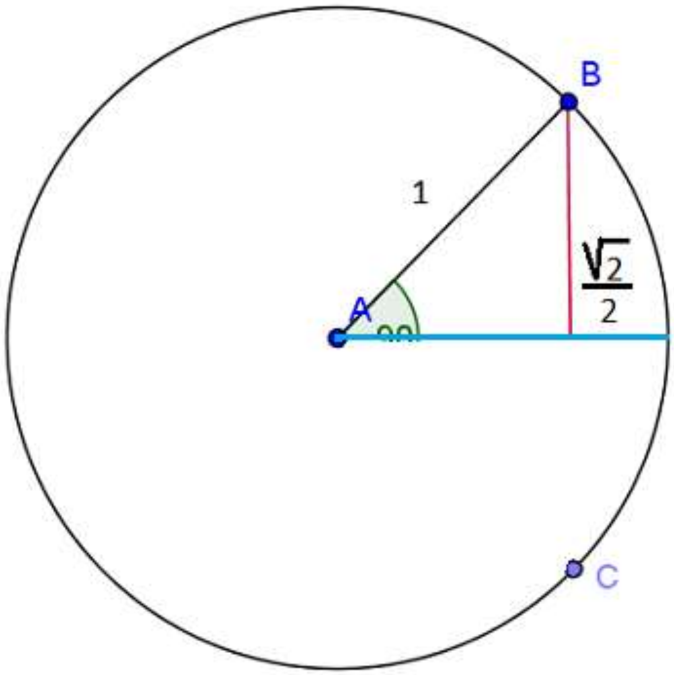
The length of the chord associated with an angle of 60 degrees is not too difficult to find (see picture below). The angle and the chord form a triangle. The triangle must be isosceles since two of the sides are radii of the same circle. That means that the angles that are not labeled in the picture must both be equal. Because we already have a 60 degree angle, these angles have to both be 60 degrees so that the angles of the triangle will add up to 180 degrees. If all the angles are equal the triangle must be equilateral (all sides are equal). The length of the chord is 1 unit:



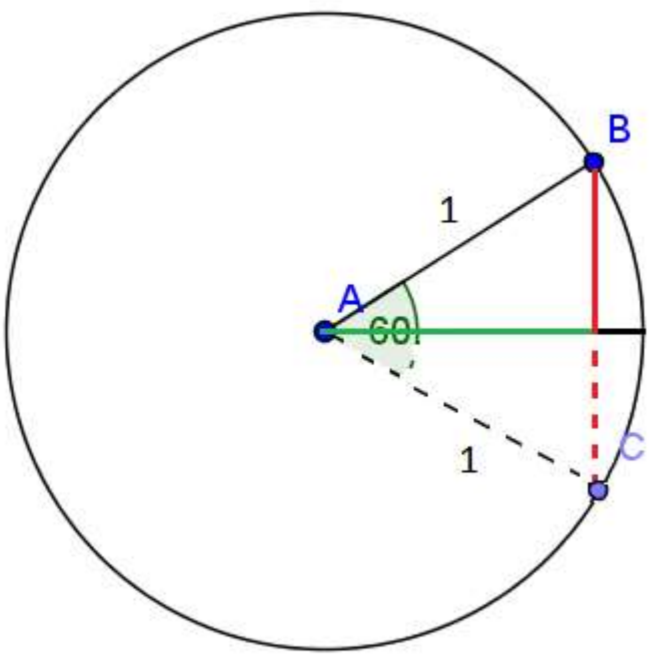
The Pythagorean Theorem helps us find the chord length for a 90 degree angle. Since the chord is the hypotenuse for the 90 degree isosceles triangle, we find c by using
$a^2 + b^2 = c^2$ or $1^2 + 1^2 = c^2$:



Eventually people make things more efficient and more standardized. Notice that I drew all of these angles facing to the right, because that eventually became the standard. Also, instead of working with chords mathematicians started working with half-chords that eventually came to be called sines (apparently due to a mistake in translating the Arabic word for "half-chord"). Correspondingly, mathematicians also considered only half the angle. The next picture shows the half-chord, or the sine, of 45 degrees ($\pi/4$ radians):



Looking at things this way allows us to introduce a second measurement; the cosine:



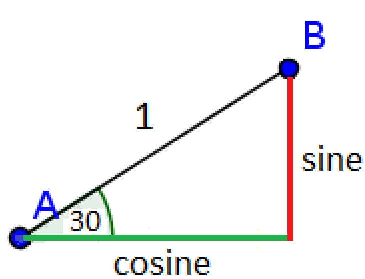
Here the solid red line shows the sine of 30 degrees, and the green line shows the cosine. The red line and the green line intersect at a right angle. We can still calculate the distances from the original chord picture using the dashed lines. The solid red line has a length of ½. Using the Pythagorean Theorem, we see that the sine ^{2} + the cosine ^{2} = 1, or $(1/2)^2$ + cosine ^{2} = 1. This means that the cosine of a 30 degree angle is the square root of ¾, or $\sqrt{3}/2$.

Because the radius of the unit circle is 1, the relationship between the sine and the cosine is always "sine ^{2} + cosine ^{2} = 1". The sine and cosine depend on an angle, so we should write this important *trigonometric identity* as:

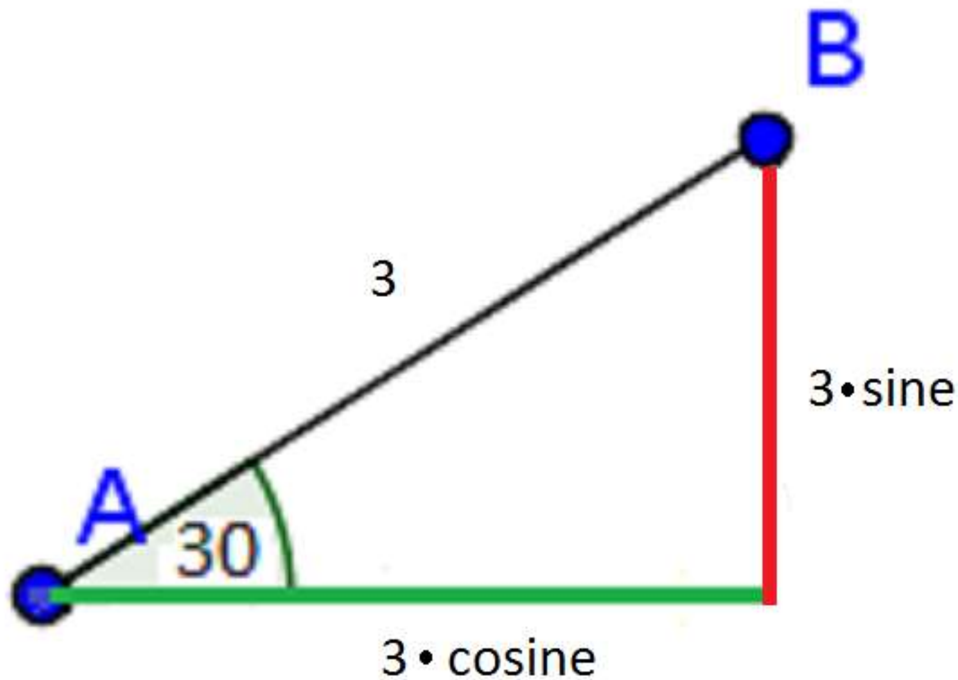
$\text{sine}^2 x + \text{cosine}^2 x = 1$.

This says that (the sine of angle x)^2 + (the cosine of angle x)^2 = 1. If you know the sine of any angle you can calculate the cosine, and vice versa.

It didn't take long for someone to take the triangle made up of the sine, the cosine and the radius, and lift it right out of the unit circle:



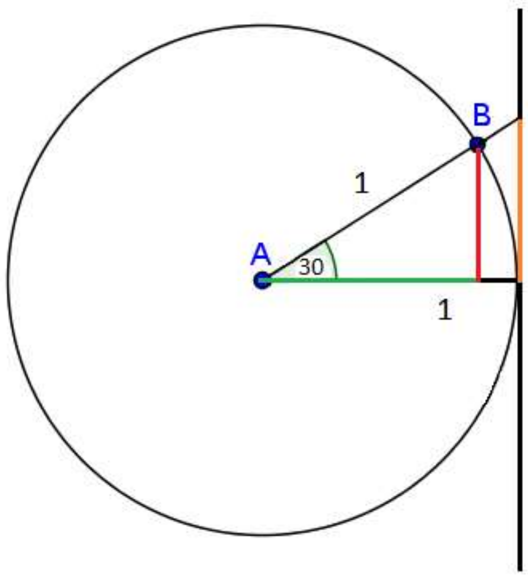
Notice that the angle at the bottom right corner of this triangle is a 90 degree angle, making it a right triangle. Knowing the sine and cosine of various angles is useful for finding the length of the sides of a *right triangle* (sometimes students will mistakenly apply the same techniques to other types of triangles, with predictably poor results). Of course not all right triangles have a hypotenuse of 1, so we may have to enlarge our picture a little:



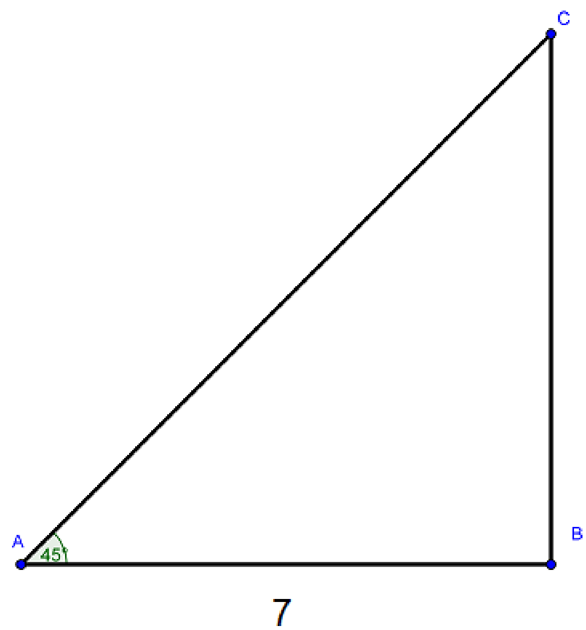
Now we have a right triangle with a hypotenuse of 3. The sides have been correspondingly enlarged to be 3 times the sine of 30 degrees, and 3 times the cosine of 30 degrees. This leads us to the idea that in any right triangle, the sine and cosine can be found by dividing the sides by the length of the hypotenuse. The sine of 30 degrees is the side opposite the angle divided by the hypotenuse, and the cosine is the side adjacent to the angle divided by the hypotenuse. Tables listing the value of the sine and cosine for different angles were created a very long time ago, and today we can get these values from a calculator. Sine and cosine are usually abbreviated as sin and cos. We have already calculated the sine and cosine of 30 degrees, so for this triangle we can say that the red side has a length of 3 times the sine of 30 degrees = 3 times $\frac{1}{2}$ = 1 ½.

The green side has a length of 3 times $\sqrt{3}/2$, which is about 2.6.

Another measurement that came from the unit circle is the tangent. At first this was an actual tangent (a tangent is a line that touches a curve at exactly one point). In the picture below, a line has been drawn tangent to the unit circle. "The tangent" of 30 degrees is defined as the length of the orange line segment.



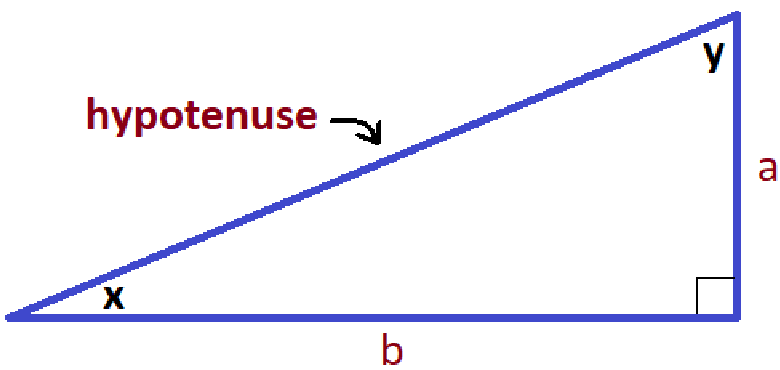
First consider the larger triangle with the orange line as one of its sides. The bottom side of the triangle is a radius of the circle, so its length is 1. We could say that the orange line divided by the bottom side is the tangent ÷ 1 = the tangent. Now consider the smaller triangle with the red side and the green side. Because its angles are the same as the angles of the larger triangle, we can say that the red side ÷ the green side = the tangent ÷ 1. That means that the sine ÷ the cosine = the tangent. If you are working with a right-angled triangle rather than in the unit circle, you can find the tangent by dividing the side opposite the angle by the side adjacent to the angle. This gives you the same ratio as sine/cosine. Because this is a ratio, the length of the hypotenuse doesn't matter at all. Your calculator will supply you with the value of the tangent (abbreviated as tan) of different angles, and you can use these values to figure out the lengths of the sides of various right triangles when some measurements are given.



For this right triangle we can use the tangent to find the length of side BC, given that AB is 7 units, and the angle at A is 45 degrees. Don't rush to grab your calculator to find the tangent of 45 degrees. Instead, draw a 45 degree angle in a unit circle, and figure out the values of the sine and the cosine. Then you will know what the tangent is. After you find the length of side BC, you could use the Pythagorean Theorem to determine the length of AC, but you should also try using the sine or cosine. [Tangent of 45° = $\frac{BC}{AB}$, sine of 45° = $\frac{BC}{AC}$, and cosine of 45° = $\frac{AB}{AC}$]

Right triangles are not always drawn in this same convenient position with the known angle in the bottom left corner. I used to turn the paper to adjust them, but I can't do that so easily on my computer. Here is where it might help you to use the mnemonic "SOHCAHTOA". You'll have to say that out loud a few times to remember it. It stands for Sine is Opposite ÷ Hypotenuse, Cosine is Adjacent ÷ Hypotenuse, Tangent is Opposite ÷ Adjacent. Opposite means the side opposite the angle. The adjacent side is the one touching the angle. Never forget that these ratios work only for *right* triangles!

The sine of an angle is equal to the cosine of the complementary angle. In the image below, x and y are complementary angles ($x + y = 90^o$):

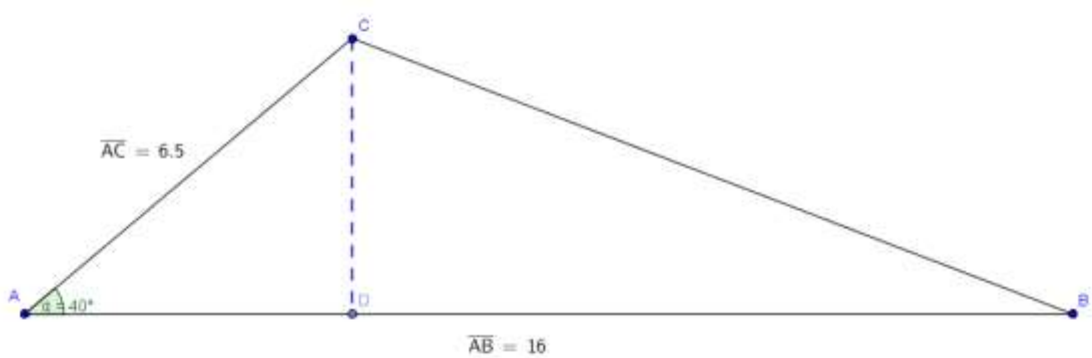


The sine of angle x is the opposite side, a, divided by the hypotenuse. The cosine of angle x is side b divided by the hypotenuse. That works the other way for angle y: The sine of angle y is side b divided by the hypotenuse, and the cosine of y is side a divided by the hypotenuse:

sin x = cos y, and cos x = sin y

Assignments

1. Because the line that measures the height of a random triangle makes a 90 degree angle with the base, we can use trigonometry to find the area of a triangle even if the height is unknown. Remember that a triangle is completely determined if you know the length of two of the sides and the measure of the angle between those sides (Side-Angle-Side, see “What is an Isosceles Triangle?”).



For the triangle shown above, we know the measures of two of the sides and the included angle. The area of the triangle is known, so we should be able to find it. Use the sine of 40 degrees to find the height CD. Show the calculations needed to find that the area of this triangle is approximately 33.4 square units.

2. Draw a right triangle that has sides of 3, 4, and 5 inches. Label the smallest angle as 1, and the next largest angle as 2. What is the sum of angles 1 and 2? Find the sine of angle 1, and the cosine of angle 2. Next, find the cosine of angle 1 and the sine of angle 2. Why do you get equal values, and why would you expect this to work out the same for any right triangle?

3. Go back to the experiment "The Mysterious Number Pi". Provide illustrations and calculations to show that when we use a regular duodecagon (a polygon with 12 sides) to estimate the value of π, we find that it must be between 3.1058 and 3.2154.

Geometry Definitions, Theorems and Postulates

Definitions

Please note that ∠ A refers to the shape of angle A, while m∠ A indicates its degree measure.

Angle Bisector: A ray that divides an angle into two congruent angles.
Betweenness of Points: Point B is between points A and C if and only if A, B, and C are collinear and AB + BC = AC. (See also: Segment Addition Postulate)
Betweenness of Rays: Ray OB is between ray OA and ray OC if and only if the measure of AOB + the measure of BOC is equal to the measure of AOC. (See also: Angle Addition Postulate)
Bisect: To divide into two equal parts
Complementary Angles: If ∠ A and ∠ B are complementary, their measures add to 90 degrees. $m\angle A + m\angle B = 90^{o}$.
Congruent Angles: Angles that have the same measure. If ∠ A ≅ ∠ B, then m∠ A = m∠ B.
Congruent Figures have the same size and shape.
Congruent Segments are segments that have the same length.
Congruent Triangles have three pairs of congruent corresponding sides and three pairs of congruent corresponding angles.
Kite: A quadrilateral with two pairs of adjacent congruent sides.
Legs of an Isosceles Triangle: The two congruent sides. The third side is the **base**.
Linear Pair: Two *adjacent* angles that are supplementary.
Midpoint of a Segment: A point that divides a segment into two congruent segments. A line segment has exactly one midpoint.
Opposite Rays: Two collinear rays with the same endpoint.
Parallelogram: A quadrilateral with parallel opposite sides.
Perpendicular Lines: Lines that meet or intersect at right angles.
Rhombus: A quadrilateral with four sides of the same length.
Skew Lines: Lines in space that are not parallel and do not intersect. Non-coplanar lines.
Supplementary Angles: If ∠ A is supplementary to ∠ B, then $m\angle A + m\angle B = 180^{o}$

Properties of Equality

These properties apply to numbers and numerical quantities, like the length of a segment or the degree measure of an angle. You can use them to justify algebraic operations like subtracting something from both sides of an equation, or dividing both sides of an equation by the same number.

Addition Property of Equality	If $a = b$, then $a + c = b + c$
Subtraction Property of Equality	If $a = b$, then $a - c = b - c$
Multiplication Property of Equality	If $a = b$, then $a \cdot c = b \cdot c$
Division Property of Equality	If $a = b$, then $a \div c = b \div c$ $(c \neq 0)$
Reflexive Property of Equality	$a = a$
Symmetric Property of Equality	If $a = b$, then $b = a$
Transitive Property of Equality	If $a = b$ and $b = c$, then $a = c$
Substitution Property of Equality	If $a = b$, then b can replace a in any expression

The *Distributive Property* says that $a(b + c) = ab + ac$. It is also used to justify adding like terms, like $2x + 3x = 5x$. That works because you can write $2x + 3x$ as $(2 + 3)x = 5x$.

Properties of Congruence

(These properties refer to congruent angles or shapes)

Reflexive Property of Congruence	$A \cong A$
Symmetric Property of Congruence	If $A \cong B$, then $B \cong A$
Transitive Property of Congruence	If $A \cong B$ and $B \cong C$, then $A \cong C$

General Postulates

Through any two points there is exactly one line.
If two lines intersect, they intersect in exactly one point.
If two planes intersect, they intersect in exactly one line.
Through any three non-collinear points there is exactly one plane.
From a given point, one and only one perpendicular can be drawn to a line.
Through a point not on a line, exactly one line can be drawn parallel to the first line.

Specific Postulates and Theorems

Angle Addition Postulate (Betweenness of Rays) If Point B is in the interior of angle AOC, then *m(the measure of)* angle AOB + *m* angle BOC = *m* angle AOC.

Angle Bisector Theorem Any point located on the bisector of an angle is at equal distance from both sides. The converse is also true. You can prove this by showing that there are AAS congruent triangles involved.

Alternate Exterior Angles Theorem If a transversal intersects two parallel lines, alternate exterior angles are congruent. The converse is also true.

Alternate Interior Angles Theorem If a transversal intersects two parallel lines, alternate interior angles are congruent. The converse is also true.

Congruent Complements Theorem If two angles are complements of the same angle, or of congruent angles, then they are congruent.

Congruent Supplements Theorem If two angles are the supplements of the same angle, or of congruent angles, then they are congruent.

Corresponding Angles Postulate If a transversal intersects two parallel lines, corresponding angles are congruent. The converse is also true.

CPCTC Corresponding Parts of Congruent Triangles are Congruent.

Exterior Angle Theorem The exterior angle of a triangle is greater than either remote interior angle.

Hinge Theorem If two triangles have two congruent sides but the included angle is larger in the second triangle, then the second triangle will have a longer third side than the first.

Hypotenuse-Leg Theorem If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the two triangles are congruent. (Pythagorean Theorem)

Isosceles Triangle Theorem If two sides of a triangle are congruent, then the angles opposite those sides (the base angles) are congruent. The converse is also true.

Parallel Theorems a) If two lines are parallel to the same line, then they are parallel to each other. **b)** In a plane, if two lines are perpendicular to the same line then they are parallel to each other (proof: angle 1 and angle 2 are right angles by the definition of perpendicular. Since the corresponding angles are congruent, the lines are parallel). Use this to construct two parallel lines.

Parallelograms A parallelogram is a quadrilateral with parallel opposite sides. Opposite sides are of equal length, and opposite angles are equal. The diagonals of a parallelogram bisect each other.

– A shape with 4 equal sides (a rhombus) is a parallelogram
– A quadrilateral with two opposite parallel sides of equal length is a parallelogram
– If a quadrilateral has opposite sides of equal length, it is a parallelogram
– If opposite angles of a quadrilateral are equal, it is a parallelogram
– If the diagonals of a quadrilateral bisect each other, it is a parallelogram

Perpendicular Bisector Theorem If a point is on the perpendicular bisector of a segment, then it is equidistant from both endpoints of the segment. The converse is also true.

Perpendicular Transversal Theorem In a plane, if a line is perpendicular to one of two parallel lines, then it is also perpendicular to the other.

Protractor Postulate Let OA and OB be opposite rays in a plane. OA and OB, and all the rays with endpoint O that can be drawn on one side of line AB can be paired with the real numbers from 0 to 180 so that 1: OA is paired with 0 and OB is paired with 180, and 2: If ray OC is paired with x and ray OD is paired with y then the measure of angle COD is $|x - y|$.

Right Angle Theorems a) All right angles are congruent. **b)** If two angles are congruent and supplementary, then each is a right angle.

Ruler Postulate The points on a line can be put into one-to-one correspondence with the real numbers so that the distance between any two points is the absolute value of the difference of the corresponding numbers.

Same – Side Interior Angles Theorem If a transversal intersects two parallel lines, same-side interior angles are supplementary. The converse is also true.

Segment Addition Postulate (Betweenness of Points) If A, B and C are collinear points, and B is between A and C, then AB + BC = AC. (AC refers to the length of segment AC).

Side-Splitter Theorem If a line parallel to one side of a triangle intersects the other two sides, then it divides the sides proportionately. Corollary: if three parallel lines intersect two transversals, then the intercepted segments are proportional.

Third Angles Theorem If two angles of one triangle are congruent to two angles of another triangle, then the third angles are also congruent.

Triangle-Angle Bisector Theorem If a ray bisects an angle of a triangle, then it divides the opposite side into two segments that are proportional to the other two sides.

Triangle Angle-Sum Theorem The sum of the measures of the angle of a triangle is 180 degrees.

Triangle Congruence SSS, SAS, AAS, ASA

Triangle Exterior Angles Theorem The measure of each exterior angle of a triangle equals the sum of the measures of the two remote interior angles.

Triangle Midsegment Theorem If a line segment joins the midpoints of 2 sides of a triangle, then that segment is parallel to the third side, and half as long.

Triangle Similarity AA Similarity, SSS Similarity. SAS Similarity: If an angle of one triangle is congruent to the corresponding angle of another triangle and the lengths of the sides including these angles are in proportion, then the triangles are similar. Side Proportionality: If two triangles are similar, the corresponding sides are in proportion.

Vertical Angles Theorem Vertical angles (opposite angles formed by the intersection of two lines) are congruent.

LOGIC

An example of a logic statement would be:
If it is a carrot, then it is orange.

The converse of this statement is:

If it is orange, then it is a carrot
The converse of a statement is not necessarily true.

The original statement has the general form **if** p, **then** q. [p → q]
The converse has the general form **if** q, **then** p. [q → p]

Statement: If it is a triangle, then it has three angles.
Converse: If it has three angles, it is a triangle.

Assignment: Create an "if…, then…" statement that has a true converse, and another "if…, then…" statement that has a converse that is not always true.

Contrapositive: **if not** q **then not** p. [~q → ~p]
If it is not orange, then it is not a carrot.

Write the contrapositive for your statements.

Inverse: **if not** p, **then not** q. [~p → ~q]
If it is not a carrot, then it is not orange (not necessarily true).

Negation: p **and not** q
It is a carrot, and it is not orange.

The converse and the contrapositive are the most popular on standardized tests and final course exams.

You can think of the converse as "flipping the statement around". If a statement and its converse are both true the statement usually makes a good definition. Statements that have a converse that is always true can be written as a biconditional: An angle is a right angle if and only if it is 90 degrees. "If and only if" is sometimes abbreviated as "iff"

The contrapositive is like a check of the original statement (q didn't happen, so p shouldn't have happened either). The contrapositive has the same truth value as the original statement. This fact seems to be quite important to people who write tests, so make sure you understand it. If the *then* part isn't true, the *if* part is not true either. Make up your own examples, and see if you can find any case where the contrapositive contradicts the original statement.

The converse and the inverse of a statement have the same truth value. If it is orange, then it is a carrot is not true because there could be some exception, like say a goldfish. If it is not a carrot, then it is not orange is not true for the same reason – again the goldfish is an exception. If there are no exceptions both the converse and the inverse are true. There are no exceptions to if it is a right angle, it is 90 degrees. So, if it is 90 degrees then it is a right angle, and if it is not a right angle then it is not 90 degrees are both true.

Law of Detachment: If $p \rightarrow q$ is a true conditional statement and p is true, then q is true.

Law of Syllogism: If $p \rightarrow q$ and $q \rightarrow r$ are true conditional statements, then $p \rightarrow r$ is true.

Two–Column Proofs

Example 1

Bob owns a small jewelry repair shop. In the evening he locks up his shop and goes home. When he returns the next day he finds that the door is open and it looks damaged. A large amount of jewelry is missing. Bob calls the police.

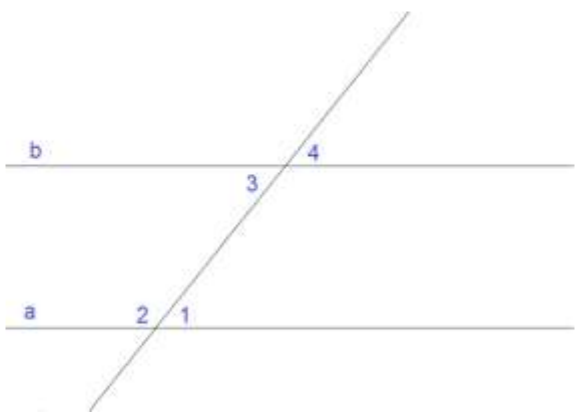
Inspector Mike "two column proof" Pearson is assigned to the case. Inspector Pearson's report reads as follows:

1. The door was locked last night	Given
2. The door is now open and damaged	Given
3. An unauthorized person entered	Forced Entry Postulate
4. Valuable stuff is missing	Given
5. An unauthorized person took it	Theft Theorem (3, 4)
6. Bob was robbed!	Definition of robbed

Important features of two-column proofs:

- Given items are put first or as close to the top as possible
- Statements are placed in logical order
- Every statement has a reason
- If two statements are needed for a particular conclusion they can be referenced by number
- The thing you are trying to prove comes last

Example 2: Alternate Interior Angles Theorem

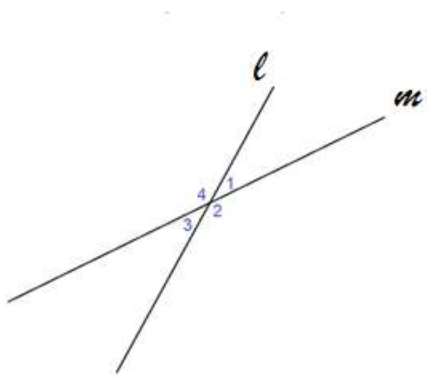


Given: Line *a* is parallel to line *b*. Prove that $\angle 1 \cong \angle 3$

Statements	Reasons
1. a ll b	Given
2. $\angle 1 \cong \angle 4$	Corresponding Angles Postulate (If two lines are parallel, then

	corresponding angles are congruent
3. $\angle 4 \cong \angle 3$	Vertical angles are congruent
4. $\angle 1 \cong \angle 3$	Transitive property of congruence

Example 3 – vertical angles



ℓ and m are two intersecting lines. Prove that angle 1 and angle 3 are congruent

Statements	Reasons
1. $\angle 1$ and $\angle 2$ are a linear pair	Given (shown in figure)
2. $\angle 1$ and $\angle 2$ are supplementary	Definition of linear pair
3. $m\angle 1 + m\angle 2 = 180$ degrees	Definition of supplementary
4. $\angle 3$ and $\angle 2$ are a linear pair	Given
5. $\angle 3$ and $\angle 2$ are supplementary	Definition of linear pair
6. $m\angle 3 + m\angle 2 = 180$ degrees	Definition of supplementary
7. $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 2$	Substitution property of equality
8. $m\angle 2 = m\angle 2$	Reflexive property of equality

9. $m\angle 1 = m\angle 3$ Subtraction property of equality
10. $\angle 1 \cong \angle 3$ Definition of congruence

Now that you have proven that vertical angles are congruent, you can use the result simply as the Vertical Angles Theorem when you need it in some other proof.

SUMMARY

The Magic Triangle: The sum of the angles inside a triangle is 180 degrees.

“Vertical” Angles: When two lines cross they form pairs of opposite angles. These “vertical” angles are congruent.

Alternate Interior (“Z”) Angles: Parallel lines create corresponding angles and alternate interior angles that are equal. Also, if those angles are equal, the two lines are parallel.

What is an Isosceles Triangle?: If a triangle has two equal sides the base angles are equal. If you pick the length of two sides of a triangle, and choose an angle between those sides, you have completely determined the shape of that triangle (SAS congruence).

Angles of a Quadrilateral: The sum of the angles inside a quadrilateral is 360 degrees. For quadrilaterals with two parallel sides, the same-side interior angles add to 180 degrees.

Angles of a Polygon – Interior: The sum of the angles of a polygon is found by dividing the polygon up into triangles. An n-sided polygon can be divided into n – 2 triangles. Therefore, the sum of the angles is 180(n – 2). To find an individual angle of a regular polygon, divide the sum of the angles by how many angles there are.

Angles of a Polygon – Exterior: An interior (inside) angle and its exterior (outside) angle add to 180 degrees. The exterior angles of a polygon always add up to 360^{o}.

The Sides of a Triangle: It is not always possible to create a triangle using three random line segments. One segment cannot be longer than the sum of the other two (Triangle Inequality). If two triangles have sides with the same lengths they are congruent (SSS congruence).

Angles and Sides - Which One Goes Where?: The largest angle of a triangle is opposite the longest side, and the smallest angle is opposite the shortest side. An exterior angle of a triangle is equal to the sum of the two opposite angles

The Pythagorean Theorem: $a^2 + b^2 = c^2$. A 45-45-90 triangle has sides x, x, and a hypotenuse of $x\sqrt{2}$, according to the Pythagorean Theorem. A 30-60-90 triangle is half of an equilateral triangle. It has sides x, $x\sqrt{3}$, and a hypotenuse of 2x.

@$$ Congruence?: There is SSS, SAS, and ASA or AAS congruence, but no SSA congruence (it depends on the size of the angle). The known angle must be between the two known sides. The Hypotenuse-Leg Theorem is really SSA congruence for right triangles.

Where is the Middle? You can find the midpoint of a line segment by taking the average of the x-coordinates and the average of the y-coordinates of the two endpoints. The coordinates of the midpoint are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. You can use the Pythagorean Theorem to determine the distance between any two points in a coordinate system.

Are Triangles with the Same Angles the Same? Triangles with the same angles have AAA Similarity, and their sides are in the same ratio.

How to Shrink a Triangle: When you shrink two of the sides of a triangle by the same factor, and leave the top angle the same, you create a similar triangle that has the same angles as the original. This principle is called SAS similarity. All similar figures have sides that are in the same ratio.

More Similarity: Splitting Your Sides The Side Splitter Theorem says that if a line parallel to one side of a triangle intersects the other two sides, then it divides the sides proportionately. If three parallel lines intersect two transversals, then the intercepted segments are proportional.

Trapezoids: Chopping a Triangle. A line drawn through the middle of a trapezoid, parallel to the bases, has a length equal to the average length of the bases.

Tangled Triangles: If a right triangle is divided into two smaller right triangles, all three triangles are similar and we can use proportions to solve for unknown sides.

Thales and the Pyramid: Thales cleverly used proportions to determine the height of the Great Pyramid.

Compass Construction: A Perpendicular Line. Learn how to use your compass to help you draw a perpendicular line.

What Makes a Parallelogram? 1. All sides are equal (a rhombus). 2. One pair of opposite sides is parallel and equal in length. 3. One pair of opposite sides consists of line segments equal in length to each other, and another pair of opposite sides also consists of segments of equal length. 4. Same-side interior angles add to 180 degrees. As a result, two consecutive angles of a parallelogram add up to 180 degrees. 5. The opposite angles of a parallelogram are equal, and if the opposite angles of a quadrilateral are equal it is a parallelogram. 6. The diagonals bisect each other, and if the diagonals of a quadrilateral bisect each other it is a parallelogram.

The Mysterious Number Pi: π is the number you get when you divide the circumference of a circle by the diameter. Pi can be approximated by drawing polygons inside and around the circle – the more sides your polygon has the more accurate the value will be.

Understanding Area: 1. The perimeter is the distance around the outer edge of the figure (add all the sides). 2. The area of a triangle is one half of the base times the height, because you can draw a box around it. 3. The area of a parallelogram is the base times the height. 4. The circumference of a circle is $2\pi r$ (π times the diameter), and the area is πr^2. The formula with the square in it belongs with the area, since area is measured in square units. 5. The area of a trapezoid is the average length of the two bases times the height. 6. To find the area of a regular polygon, divide it up into triangles, or multiply ½ times the apothem by the perimeter. A regular hexagon divides up into 6 equilateral triangles. You can then find the area of these triangles by using the dimensions of the special 30-60-90 triangle. 7. The area of a rhombus, a kite, or a square is one half of the product of the diagonals, because you can draw a box around them. 8. For similar 2-D figures, if the sides are in the ratio a : b, then the areas are in the ratio $a^2 : b^2$.

Going in Circles: The equation of a circle with center at the origin is $x^2 + y^2 = r^2$. If the center is at (h,k) then the equation is $(x – h)^2 + (y – k)^2 = r^2$.

How to Balance a Line on a Circle: The radius of a circle at a given point on the circle is perpendicular to a tangent line drawn at that point.

The Incenter: Putting a Circle inside Your Triangle: The incenter lies at the intersection of the angle bisectors of the triangle. This happens because any point located on the bisector of an angle is at equal distance from both sides (Angle Bisector Theorem).

The Circumcenter: Putting Your Triangle inside A Circle: The circumcenter lies at the intersection of the perpendicular bisectors. Any point on the bisector of a segment will be the same distance from both endpoints (Perpendicular Bisector Theorem).

Balancing a Triangle: The Centroid. If you draw all three medians, you divide the triangle up into 6 areas that are all equal in size, and therefore in weight. The triangle balances at the centroid. The centroid can be found along any single median by measuring out 2/3 of the distance from the vertex to the opposite side.

Finding Your Center: The Orthocenter. This is the point where the altitudes of a triangle intersect. 1. Parallel lines have the same slope. 2. Perpendicular lines have slopes that multiply to -1. 3. Use $y = mx + b$ or $y - y_1 = m(x - x_1)$ to create a line if you know the slope m and one point. 4. To find the intersect point of two lines, set their equations equal to each other.

An Angle in a Semicircle: A triangle inscribed in a semicircle must be a right triangle.

An Angle in a Circle: An angle inscribed in a circle is one-half the size of the intercepted arc. If you draw a random 4-sided polygon inside a circle so that all of the vertices are on the edge of the circle, its opposite angles will sum to 180 degrees.

A Chord and a Tangent: The angle between a chord and a tangent is one-half the size of the intercepted arc.

Intersecting Chords: If one chord can be divided into segments a and b, and the other chord consists of segments c and d, then ab = cd.

Two Angles in a Circle: When two chords intersect in a circle, the sum of the vertical angles created by the intersection is equal to the sum of the intercepted arcs.

An Angle Outside of a Circle: The measure of the angle outside the circle is one-half the difference of the measures of the intercepted arcs.

More Circle Theorems: If two line segments drawn from a point outside a circle both intersect the circle, the product of length of the first segment and its part outside the circle is equal to the product of the length of the second segment and its part outside the circle. If one or both segments are tangent to the circle, then the entire length of the segment is the same as the length of the part outside the circle.

Understanding Volume: The volume of a prism is length times width times height, which is the same as base times height. The volume of a slanted prism is calculated in the same way. A smaller shape has a relatively larger surface area than a similar bigger shape. For similar 3-D figures, if the sides are in the ratio a : b then the volumes are in the ratio $a^3 : b^3$

The Surface Area and Volume of a Cylinder: You can find the surface area of a complex figure by adding up all of the individual surface areas that it can be divided into. The surface area of a cylinder is composed of the two bases and the middle part: $2\pi r^2 + 2\pi rh$, where h is the height and r is the radius. The volume is base times height, which is $\pi r^2 h$.

The Surface Area and Volume of a Cone: The lateral surface area of a cone is given by the formula πrs, where s is the slant height of the cone. The volume of a cone is 1/3 of the volume of the cylinder that would contain it.

The Volume of a Pyramid: The volume of a pyramid is 1/3 of the volume of the prism that would contain it – 1/3 times base times height.

The Volume and Surface Area of a Sphere: The surface area of a sphere is $4\pi r^2$, and the volume is $\frac{4}{3}\pi r^3$. The formula with the square belongs with area, while the volume is measured in cubic units

Transformation Target Practice: Reflections produce a mirror image, which may be partially rotated depending on the slope of the reflection line. The only way to get an image that has the same orientation as the original image is to reflect it over two parallel lines, or twice over the same line. The object will move twice the distance between the two parallel lines.

Trigonometric Ratios: There are 2π radians in a circle, so 2π radians = 360 degrees. Sohcahtoa: sine is opposite over hypotenuse, cosine is adjacent over hypotenuse, tangent is opposite over adjacent. The sine of an angle is equal to the cosine of its complementary angle and vice versa.

Logic: Conditional statement: if it is a carrot, then it is orange. The *converse* (if it is orange, then it is a carrot) may or may not be true. The *contrapositive* is logically equivalent to the original statement: If it is not orange, then it is not a carrot. Think of this as a check on the original.